# Various Approaches to Teach Light Deflection in Gravitational Fields

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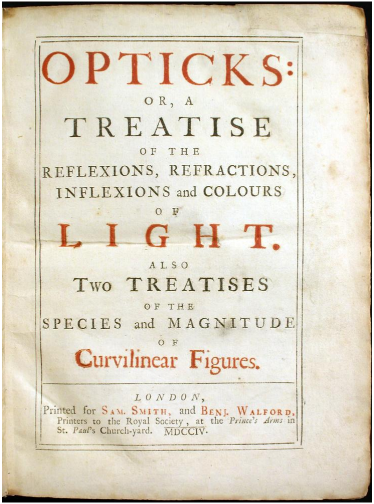
# Ballistic Light Deflection Before EINSTEIN – NEWTON


OPTICKS:
OR, A
TREATISE
OF THE
REFLEXIONS, REFRACTIONS,
INFLEXIONS and COLOURS
OF
LIGHT.
ALSO
Two TREATISES
OF THE
SPECIES and MAGNITUDE
OF
Curvilinear Figures.

*LONDON*,
Printed for SAM. SMITH, and BENJ. WALFORD,
Printers to the Royal Society, at the *Prince's Arms* in
St. *Paul's* Church-yard. MDCCIV.

ISAAC NEWTON (1704):

*„Do not Bodies act upon Light at a distance, and by their action bend its rays, and is not this action* (cateris paribus) *strongest at the least distance?“*

NEWTON, I.:
The Third Book of Opticks,
Query No. 1

# Ballistic Light Deflection Before EINSTEIN – SOLDNER

JOHANN GEORG V. SOLDNER
(1776-1833)

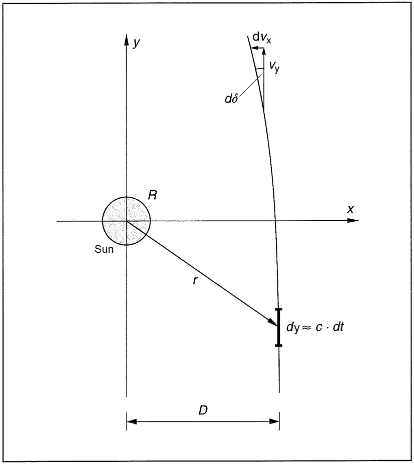
JOHANN GEORG V. SOLDNER (1801):

*„Hopefully, nobody will consider it a problem that I treat a light ray almost as if it were a heavy body . . . One cannot imagine an object that could exist and act upon our senses without having the quality of matter.“*

V. SOLDNER, J.:
Über die Ablenkung eines Lichtstrals von seiner geradlinigen Bewegung, durch die Attraktion eines Weltkörpers, an welchem er nahe vorbei geht

Astronomisches Jahrbuch für das Jahr 1804, Berlin 1801, pp. 161–172

# Ballistic Light Deflection




$M$: Mass of the Sun
$D$: Impact parameter

- Small light deflection expected:

$$x \approx D$$

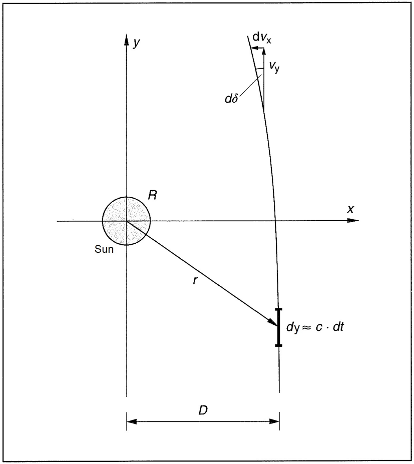
- Speed of light in $y$ direction:

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} \approx c = \mathrm{const}$$

- NEWTONs lex secunda:

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{GMx}{(x^2 + y^2)^{\frac{3}{2}}}$$
$$\approx -\frac{GMD}{(D^2 + y^2)^{\frac{3}{2}}}$$

# Ballistic Light Deflection




$\mathrm{d}\delta$: Deflection angle

- Light deflection ...

$$\tan(\mathrm{d}\delta_N) \approx \mathrm{d}\delta_N = \frac{|\mathrm{d}v_x|}{v_y} = \frac{GMD}{c^2} \frac{\mathrm{d}y}{(D^2 + y^2)^{\frac{3}{2}}}$$

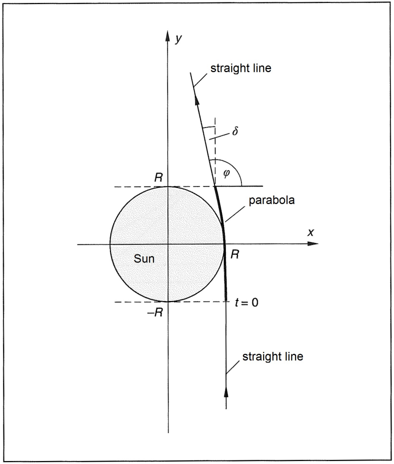
- ... for the entire light path:

$$\delta_N = \frac{GMD}{c^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{(D^2 + y^2)^{\frac{3}{2}}} = 2\frac{GM}{c^2} \cdot \frac{1}{D}$$

- Sun ($M = M_\odot$, $D = R_\odot$):

$$\delta_N = 0{,}87''$$

# Ballistic Light Deflection Simplified




$M$: Mass of the Sun
$R$: Radius of the Sun,
impact parameter $(D = R)$

- Light deflection only for

$$-R \leq y \leq +R$$

- Acceleration in $x$ direction:

$$g = \frac{GM}{R^2} = \mathrm{const}$$

- Light path: Parabola

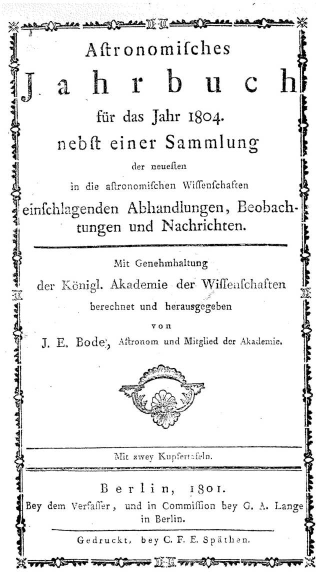
$$x = -\frac{g}{2}\frac{(y+R)^2}{c^2} + R$$

- Deflection angle:

$$\delta_N \approx \tan\delta_N = -\left.\frac{\mathrm{d}x}{\mathrm{d}y}\right|_{y=R}$$

$$= g\frac{2R}{c^2} = 2\frac{GM}{c^2}\cdot\frac{1}{R}$$

# Ballistic Light Deflection Before EINSTEIN – SOLDNER



Aſtronomiſches

Jahrbuch

für das Jahr 1804.

nebſt einer Sammlung

der neueſten

in die aſtronomiſchen Wiſſenſchaften

einſchlagenden Abhandlungen, Beobachtungen und Nachrichten.

Mit Genehmhaltung

der Königl. Akademie der Wiſſenſchaften

berechnet und herausgegeben

von

J. E. Bode, Aſtronom und Mitglied der Akademie.

Mit zwey Kupfertafeln.

Berlin, 1801.

Bey dem Verfaſſer, und in Commiſſion bey G. A. Lange in Berlin.

Gedruckt, bey C. F. E. Späthen.

JOHANN GEORG V. SOLDNER (1801):

*„Therefore, if a light ray passes by a celestial body it will be forced by the attraction of that body, instead of moving in a straight line, to carry out a hyperbola whose concave part is directed towards the attracting body . . . .“*

*„If one could observe fixed stars very close to the sun, we had to take this into consideration . . . .“*

# FERMATs Principle

The trajectory of light is the one that yields a stationary value for the **optical** path length,

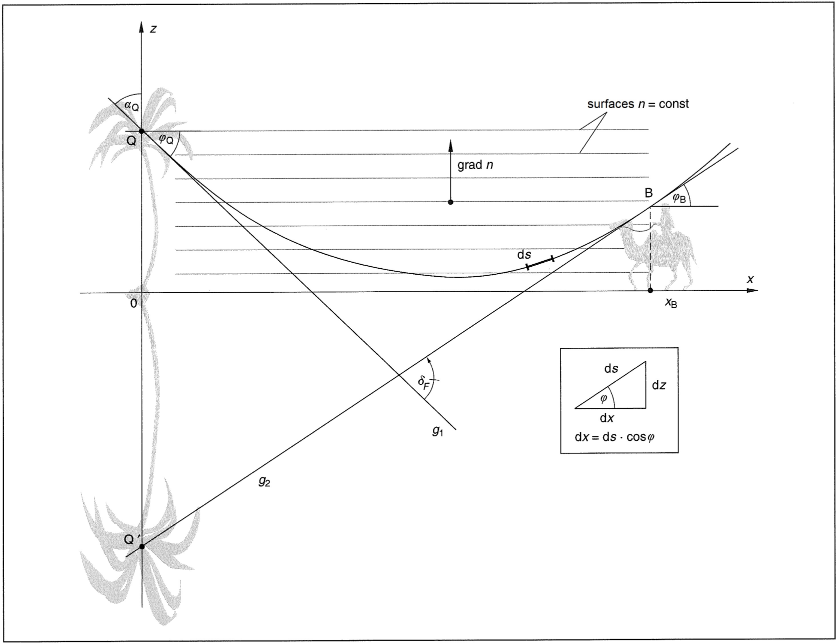
$$\int_Q^B n \, \mathrm{d}s = \mathrm{Extr.}$$

($n$: refractive index)

- EULER-LAGRANGE equations:
  (geometrical form of the light ray)

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\vec{r}}{\mathrm{d}s}\right) = \mathrm{grad}\, n$$

# Fata Morgana – Mirage on Earth



# Fata Morgana – Mirage on Earth

- Refractive index of the air:

$$n(z) = n_0\sqrt{1 + 2\frac{z}{a}}$$

$$\mathrm{grad}\, n = \frac{n_0^2}{an}\vec{e}_z$$

- EULER-LAGRANGE equation (1):

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}x}{\mathrm{d}s}\right) = 0$$

Conservation law (SNELL):

$$n\frac{\mathrm{d}x}{\mathrm{d}s} = n\cos\varphi$$
$$= n\sin\alpha = C = \mathrm{const}$$

- EULER-LAGRANGE equation (2):

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}z}{\mathrm{d}s}\right) = \frac{n_0^2}{an}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{n_0^2}{aC^2}$$

Shape of the orbit: Parabola

$$z = \frac{n_0^2}{2aC^2}\cdot x^2 + \tan\varphi_Q\cdot x + z_Q$$

- Light deflection:

$$\delta_F = \frac{n_0^2 x_B}{an_Q^2 + n_0^2 x_B\tan\varphi_Q}$$

$$\mathrm{d}s_{(4)}^2 = \frac{\mathrm{d}r^2}{1 - \frac{R_S}{r}} + r^2 \left( \mathrm{d}\vartheta^2 + \sin^2\vartheta \, \mathrm{d}\varphi^2 \right) - \left( 1 - \frac{R_S}{r} \right) c^2 \, \mathrm{d}t^2$$

. . . in **isotropic coordinates**

- New radial coordinate $\bar{r}$:

$$r = \bar{r} \left( 1 + \frac{1}{4} \frac{R_S}{\bar{r}} \right)^2$$

- Result: Conformally flat space

$$\mathrm{d}s_{(4)}^2 = \left( 1 + \frac{1}{4} \frac{R_S}{\bar{r}} \right)^4 \left( \mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 + \mathrm{d}\bar{z}^2 \right) - \frac{\left( 1 - \frac{1}{4} \frac{R_S}{\bar{r}} \right)^2}{\left( 1 + \frac{1}{4} \frac{R_S}{\bar{r}} \right)^2} c^2 \, \mathrm{d}t^2$$

# The SCHWARZSCHILD-Radius

How to express the mass $M$ of a celestial body „more geometrico“?

Schwarzschild radius:

$$R_\mathrm{S} = 2\frac{GM}{c^2}$$

Compactness:

$$\mathcal{C} \equiv \frac{R_\mathrm{S}}{R}$$

($R$: size of the body)

Examples:

- Earth: $R_\mathrm{Earth} = 0{,}89\,\mathrm{cm}$, $\mathcal{C} = 1{,}4 \cdot 10^{-9}$
- Sun: $R_\mathrm{Sun} = 2{,}96\,\mathrm{km}$, $\mathcal{C} = 4{,}3 \cdot 10^{-6}$

**From now on**:

$$\frac{R_S}{r} \ll 1$$

# Curvature of Time *And* Space Matters!

- SCHWARZSCHILD spacetime in isotropic coordinates $\left(\frac{R_S}{r} \ll 1\right)$

$$\mathrm{d}s_{(4)}^2 = \left(1 + \frac{R_S}{r}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) - \left(1 - \frac{R_S}{r}\right) c^2\,\mathrm{d}t^2$$

- Propagation of light: $\mathrm{d}s^2 = 0$

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \left(1 - 2\frac{R_S}{r}\right) c^2\,\mathrm{d}t^2$$

  - Without curvature of space (EINSTEIN 1911):

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \left(1 - \frac{R_S}{r}\right) c^2\,\mathrm{d}t^2$$

Factor 2 (!)

# The refractive index of SCHWARZSCHILD spacetime

- Speed of light:

$$\tilde{c} = \left(1 - \frac{R_S}{r}\right) c$$

  - EINSTEIN 1911:

$$\tilde{c} = \left(1 - \frac{1}{2} \cdot \frac{R_S}{r}\right) c = \left(1 + \frac{U_N}{c^2}\right) c$$

with $U_N = -\frac{GM}{r}$: NEWTONian gravitational potential

- Refractive index:

$$\tilde{c} = \frac{c}{n} \longrightarrow n(r) = 1 + \frac{R_S}{r}$$

*„The gravitational field acts upon light rays in the same way as if the Sun were surrounded by a refractive medium whose refractive exponent $n$ … varies with the distance $r$ from the Sun.“*

HERMANN WEYL: Raum – Zeit – Materie (1923)

# Fata Morgana in the Sky

- Refractive index:

$$n(r) = 1 + \gamma \frac{R_S}{r}$$

  - Soldner, Einstein 1911: $\gamma = \frac{1}{2}$
  - Einstein 1915: $\gamma = 1$

- Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}\right) = \frac{\mathrm{d}n}{\mathrm{d}r} \cdot \frac{\vec{r}}{r} \;\Big|\; \vec{r}\times$$

Conservation law . . .

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\vec{r} \times n \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}\right) = 0$$

. . . in polar coordinates

$$n(r) r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}s} = \mathrm{const}$$

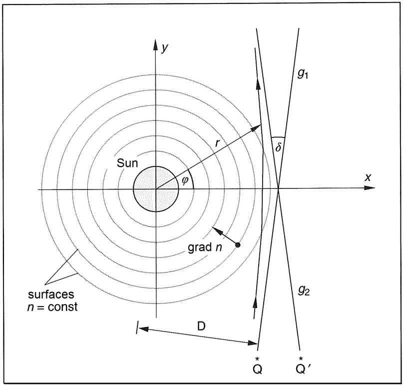
- New variable $u \equiv \frac{1}{r}$ and $A = \mathrm{const}$:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 = A(1 + 2\gamma R_S u)$$

- Second-order, linear equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = A\gamma R_S$$

# Fata Morgana in the Sky




$D$: Impact parameter ($\frac{R_S}{D} \ll 1$)
$\delta$: Light deflection
$g_1$, $g_2$: Asymptotes

- Solution: Hyperbola

$$r = \frac{\frac{D^2}{\gamma R_S}}{1 + \frac{D}{\gamma R_S}\cos\varphi}$$

- Slope of the asymptotes:

$$m_{1,2} = \pm \frac{D}{\gamma R_S}$$

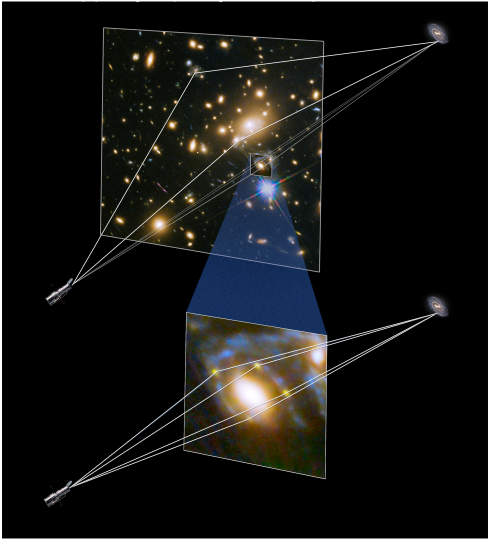
- Deflection angle:

$$\delta = 2\gamma \frac{R_S}{D}$$

- Example: The Sun ($D = R_\odot$)

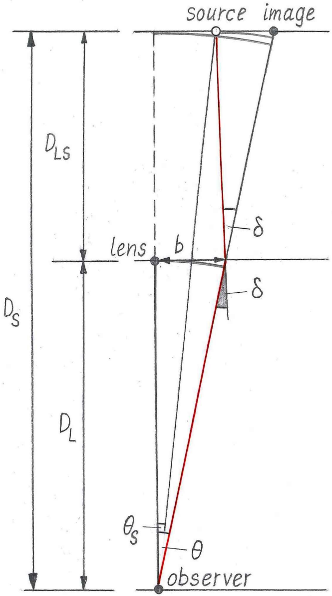
$$\delta = \begin{cases} 0{,}87'' & \text{für } \gamma = \frac{1}{2} \\ 1{,}75'' & \text{für } \gamma = 1 \end{cases}$$

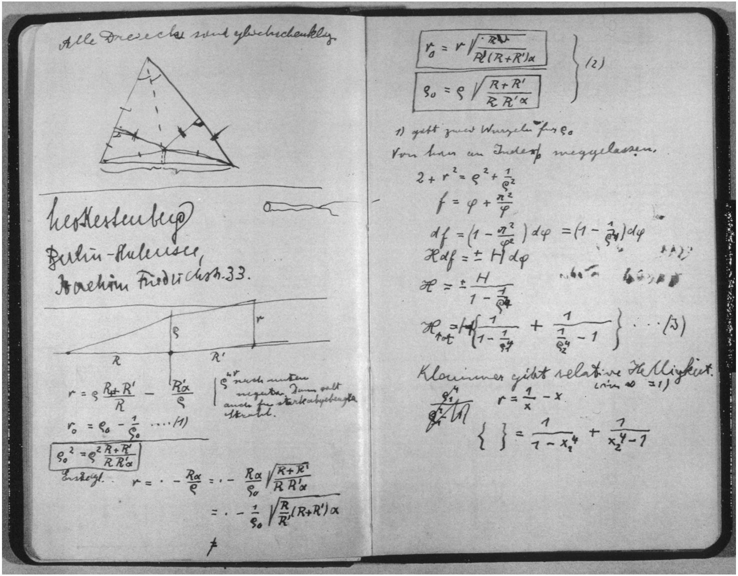
# Gravitational Lensing – What Can We Observe?



The supernova „Refsdal“

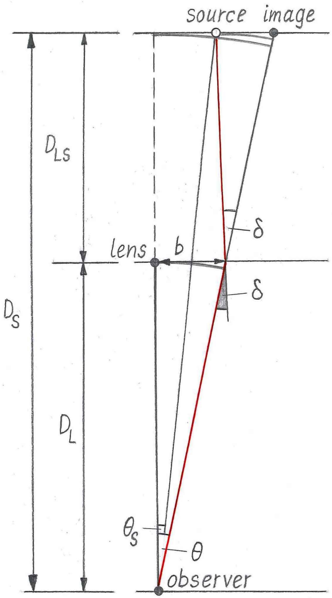
(Hubble Space Telescope: STScI-2015-08d)






Jewish National and University Library, Hebrew University of Jerusalem

# Gravitational Lensing – The Approximations




- Small-angle approximation:

$$R_S \ll b \ll D_L, D_{LS}$$

- Deflection occurs at a single point near the lens.
- b: impact parameter

$$b \approx \Theta \cdot D_L$$

- $\delta$: deflection angle

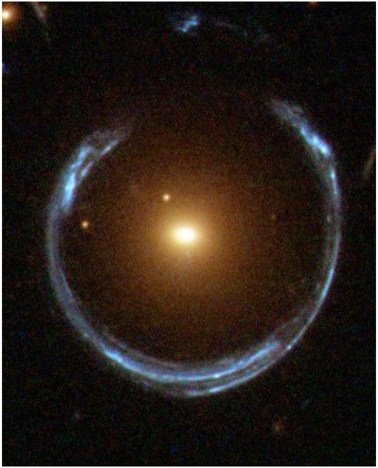
$$\delta = 2\frac{R_{S,L}}{b}$$

$R_{S,L}$: Schwarzschild radius of the lens

- From the figure:

$$\Theta D_S \approx \Theta_S D_S + \delta D_{LS}$$

# Lens Equation and EINSTEIN Radius



The „Cosmic Horseshoe“
LRG 3-757

(Hubble Space Telescope)

- **The lens equation**:

$$\Theta^2 - \Theta_S\Theta - 2R_{S,L}\frac{D_{LS}}{D_L D_S} = 0$$

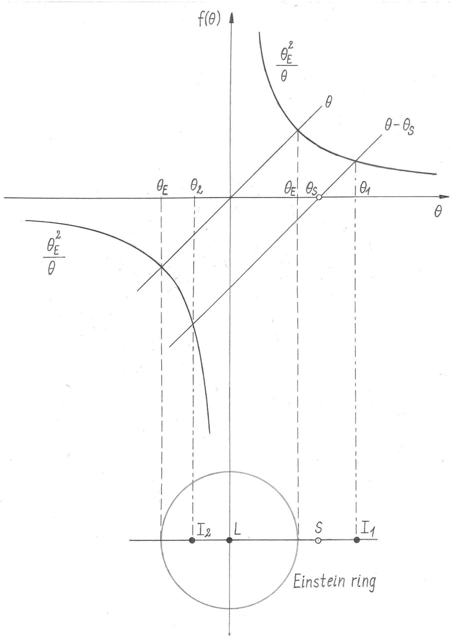
- $\Theta_S = 0$: observer, lens and source perfectly aligned
- Because of symmetry: The image is an EINSTEIN ring with angular radius

$$\Theta_E = \sqrt{2R_{S,L}\frac{D_{LS}}{D_L D_S}}$$

- Lens equation in terms of EINSTEIN radius:

$$\Theta^2 - \Theta_S\Theta - \Theta_E^2 = 0$$

# Solutions of the Lens Equation




- Solutions:

$$\Theta_{1,2} = \frac{\Theta_S}{2}\left(1 \pm \sqrt{1 + 4\frac{\Theta_E^2}{\Theta_S^2}}\right)$$

  i.e. two images $\mathrm{I}_1$, $\mathrm{I}_2$ or EINSTEIN ring

- Graphic solution:
  Lens equation in terms of EINSTEIN radius is equivalent to

$$\Theta - \Theta_S = \frac{\Theta_E^2}{\Theta}$$

# The Sun as a Gravitational Lens

**Peculiarity No.1**:
$D_L \ll D_S$, $D_{LS} \approx D_S$

- Angular radius of the Sun

$$\Theta_{\rm min} = \frac{R_\odot}{D_L} \approx 960''$$

- Einstein radius:

$$\Theta_E = \sqrt{2\frac{R_{S,\odot}}{D_L}} = 41{,}3''$$

unobservable: $\Theta_E \ll \Theta_{\rm min}$

- For $\Theta_E \ll \Theta_S$:

$$\frac{\Theta_E^2}{\Theta_S} \approx \delta$$

**Peculiarity No.2**:
$\Theta_S$ is observable.

- 1st solution of the lens equation:

$$\Theta_1 = \Theta_S + \delta$$

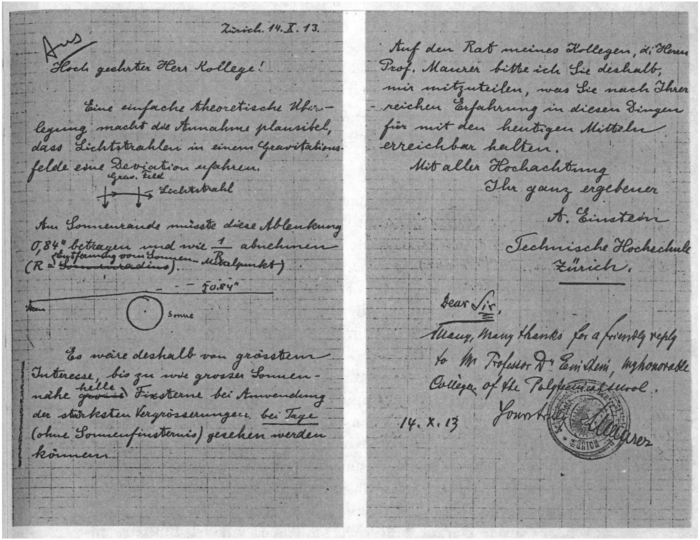
The observable angular displacement of a star is the deflection angle itself.

- 2nd solution of the lens equation:

$$\Theta_2 = -\delta$$

unobservable:
$\delta_{\rm max} = 1{,}75'' \ll \Theta_{\rm min}$

# Einsteins Letter to George Ellery Hale of October 14, 1913



Zürich. 14. X. 13.

Ans

Hoch geehrter Herr Kollege!

Eine einfache theoretische Überlegung macht die Annahme plausibel, dass Lichtstrahlen in einem Gravitationsfelde eine Deviation erfahren.



Am Sonnenrande müsste diese Ablenkung 0,84" betragen und wie $\frac{1}{R}$ abnehmen (R = ~~Sonnenradius~~ Entfernung vom Sonnen-Mittelpunkt).



Es wäre deshalb von grösstem Interesse, bis zu wie grosser Sonnennähe ~~grosse~~ helle Fixsterne bei Anwendung der stärksten Vergrösserungen bei Tage (ohne Sonnenfinsternis) gesehen werden können.

Auf den Rat meines Kollegen, d. Herrn Prof. Maurer bitte ich Sie deshalb, mir mitzuteilen, was Sie nach Ihrer reichen Erfahrung in diesen Dingen für mit den heutigen Mitteln erreichbar halten.

Mit aller Hochachtung
Ihr ganz ergebener
A. Einstein
Technische Hochschule
Zürich.

Dear Sir,

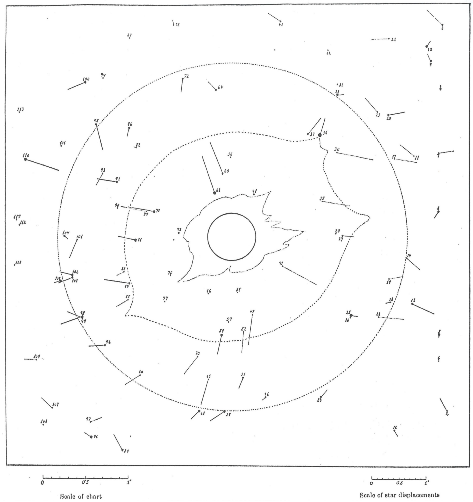
Many, many thanks for a friendly reply to Mr Professor Dr Einstein, my honorable Collega of the Polytechnical School.

14. X. 13

Your truly
Maurer

Millikan Library, California Institute of Technology

# The Total Solar Eclipse of September 21, 1922

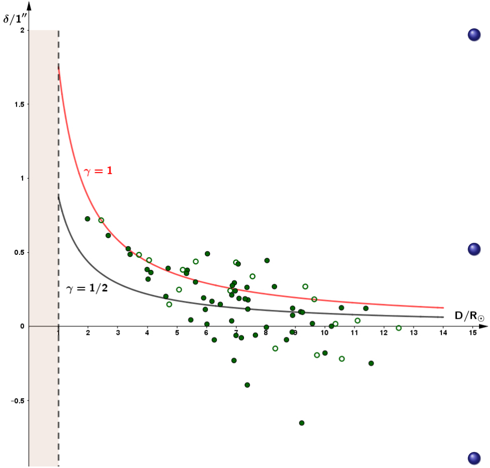



Left scale: Scale of chart (1°)
Right scale: Scale of displacements (1″)

Lick Observatory Bulletin No. 346,
July 5, 1923

- Dots ($\theta_S$):
  - Star chart of constellation Virgo
  - Picture taken at Tahiti in May 1922
- Arrows ($\theta - \theta_S$):
  - Displacement of the stars during the eclipse
  - Picture taken at Wallal (Western Australia)
  - 2250 times enlarged

# The Total Solar Eclipse of September 21, 1922




Observed radial displacements as a function of the star's distance from the Sun's center.

(Credit: Silvia Simionato)

- Dots:
  - Filled:
    High confidence level
  - Open:
    Low confidence level

  (according to the authors)
- Curves:
  - Red ($\gamma = 1$):
    Einstein 1915
  - Black ($\gamma = \frac{1}{2}$):
    Soldner,
    Einstein 1911
- Result:

$$\delta = 1{,}72'' \pm 0{,}11''$$

(Campbell, Trumpler)

# Comments on the Total Solar Eclipse of May 29, 1919

- Frank Watson Dyson:

  *„After a careful study of the plates I am prepared to say that there can be no doubt that they confirm Einstein's prediction. ... Light is deflected in accordance with Einstein's law of gravitation.“*

- Joseph John Thomson:

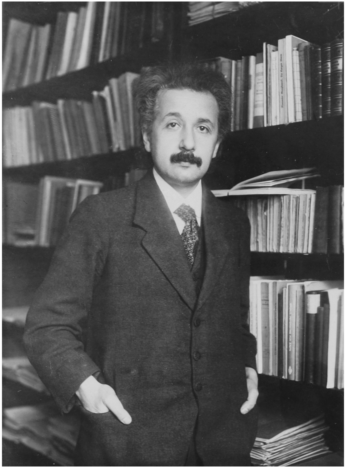
  *„This is the most important result obtained in connection with the theory of gravitation since Newton's day ... The result [is] one of the highest achievements of human thought.“*

- Ilse Rosenthal-Schneider, a student of Einstein 1919, asked him, what if there had been no confirmation of his prediction?

  Albert Einstein's response:

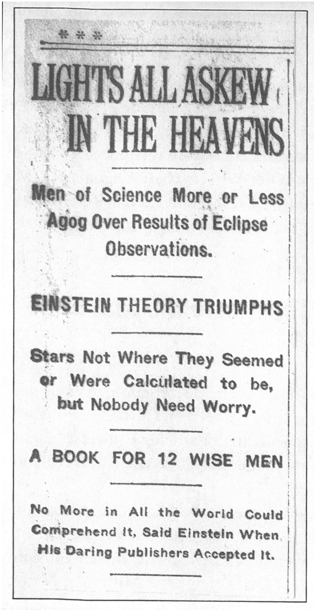
  *„Then I should have been sorry for the dear Lord; but the theory is correct.“*

# Title Page of the New York Times for November 10, 1919



ALBERT EINSTEIN
in his office, Berlin 1916

(Albert Einstein Archive, Jerusalem)



* * *

LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

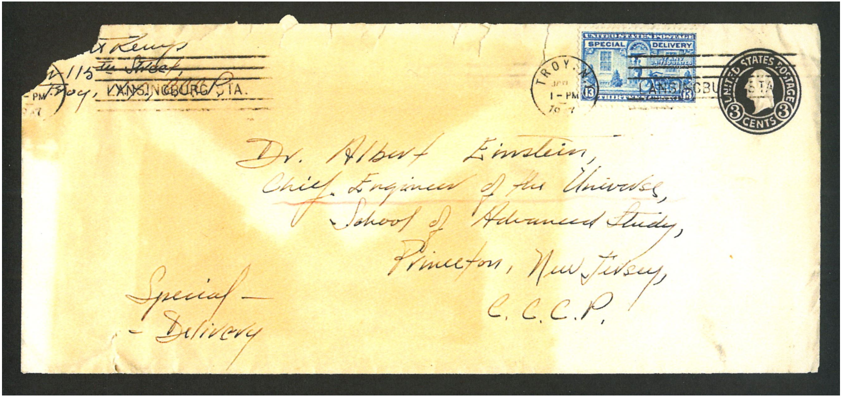
EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

# ALBERT EINSTEIN – „Chief Engineer of the Universe“




Albert Einstein Archive, Jerusalem