Combining General Relativity and Optics to Visualize Light on Curved Paths

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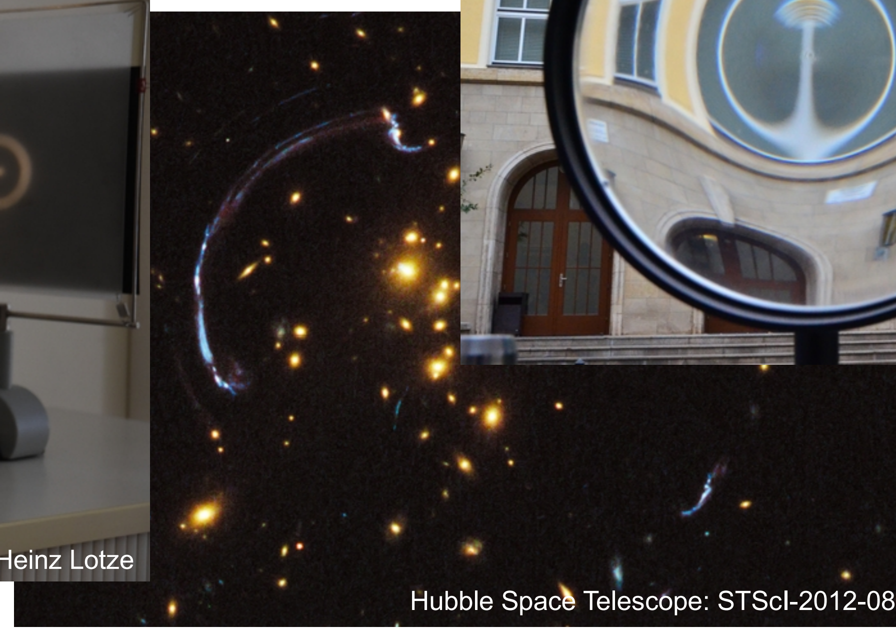
The „Cosmic Horseshoe“ LRG 3-757
(Hubble Space Telescope)

Karl-Heinz Lotze

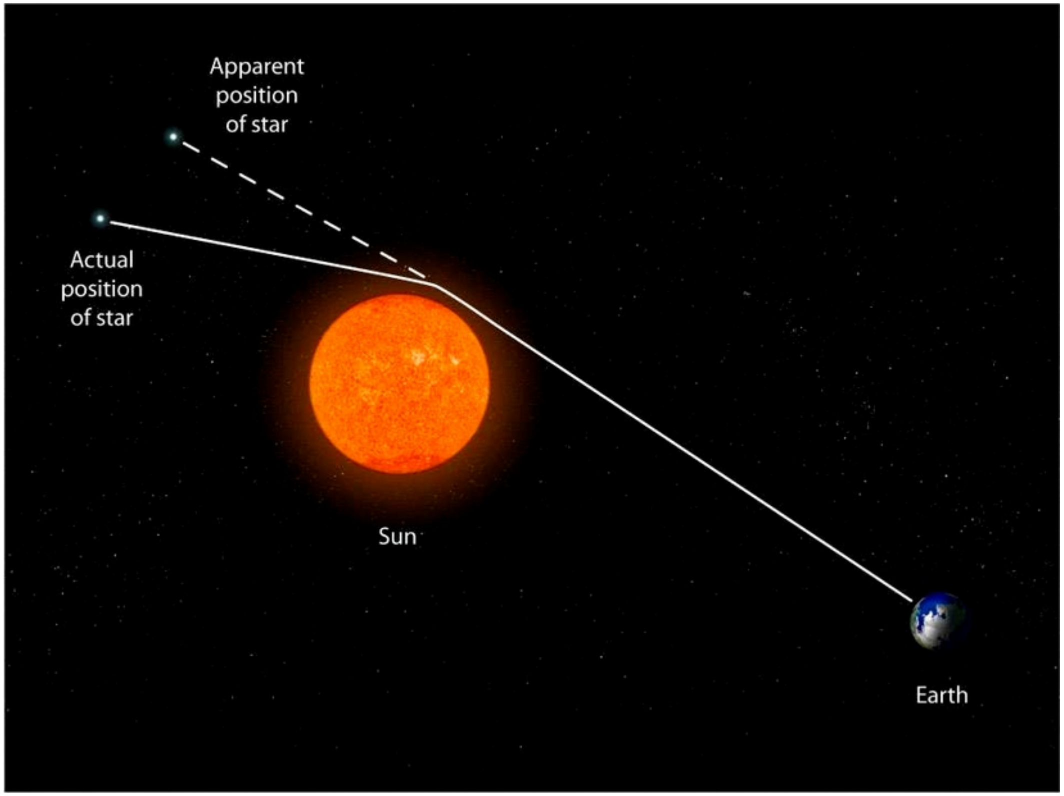


Hubble Space Telescope: STScI-2012-08

Uwe Alberti

10-15 February 2019, Physikzentrum Bad Honnef

Light Deflection



Deflection Angle

$$\boxed{\delta(\theta) = \frac{4G}{c^2} \boxed{\frac{M(\theta)}{\theta D_L}}}$$

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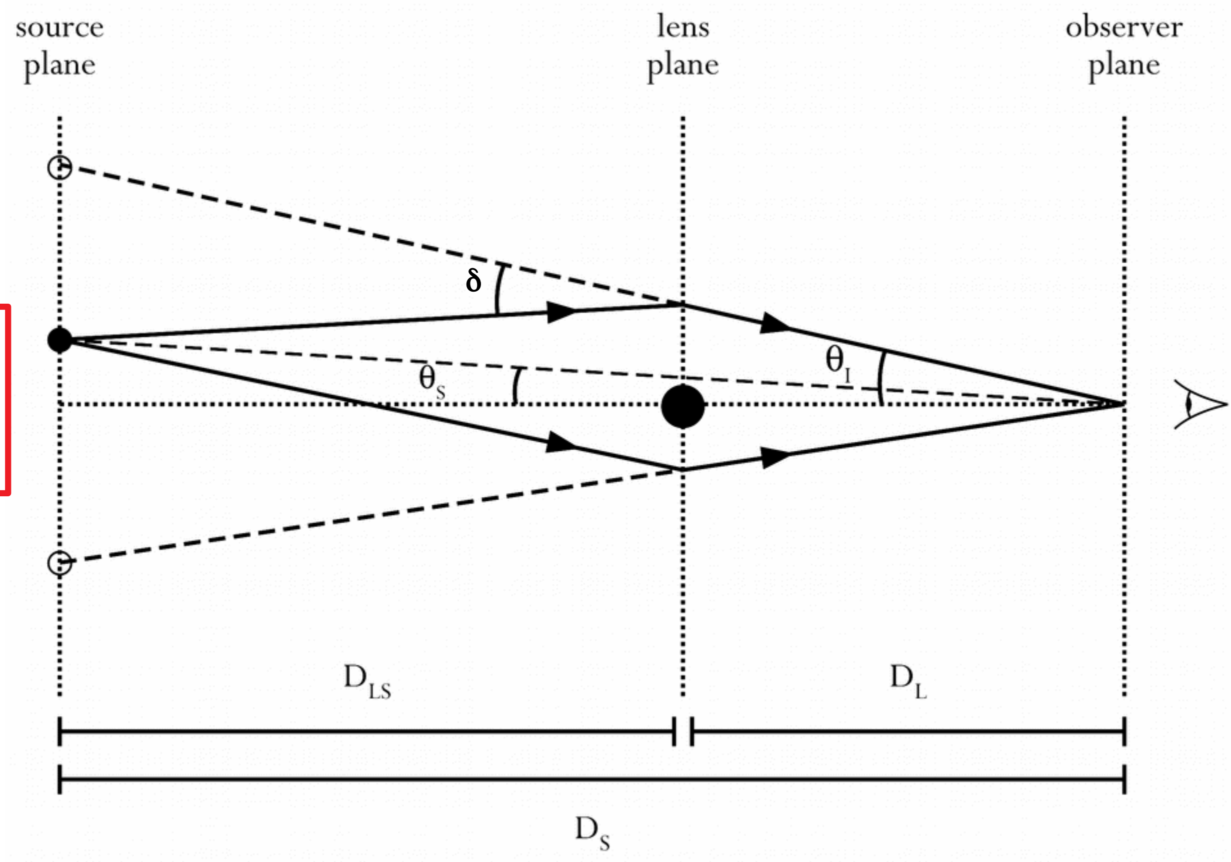
Lens Equation

- Lens Equation:

$$\theta - \theta_S - \delta \frac{D_{LS}}{D_S} = 0$$

$$\boxed{\theta - \theta_S - \frac{4G}{c^2} \frac{M(\theta)}{\theta} \frac{D_{LS}}{D_L D_S} = 0}$$



- Einstein Ring Radius:

$$\boxed{\theta_E = \frac{4G}{c^2} \frac{M(\theta_E)}{\theta_E} \frac{D_{LS}}{D_L D_S}}$$

Point-Mass Lens

$$\boxed{M = constant}$$

$$\theta - \theta_S = \frac{4 \cdot G}{c^2} \frac{M}{\theta} \frac{D_{LS}}{D_L D_S}$$

$$\theta_E = \frac{4\,G}{c^2} \frac{M}{\theta_E} \frac{D_{LS}}{D_L D_S}$$

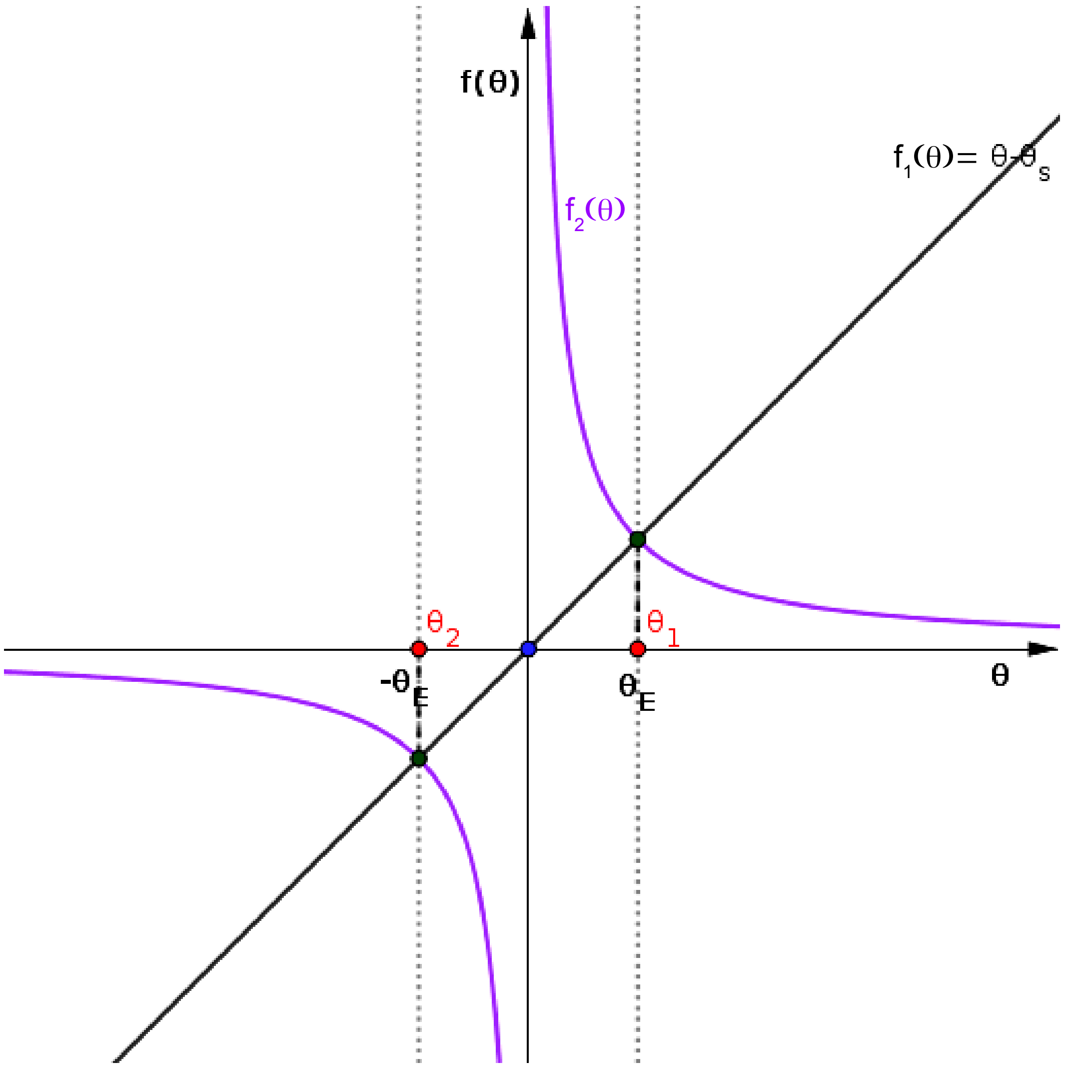
$$\boxed{\theta - \theta_S = \frac{\theta_E^2}{\theta}}$$

Lens equation in terms of Einstein radius

Point-Mass Lens

$$\boxed{\theta - \theta_S = \frac{\theta_E^2}{\theta}}$$

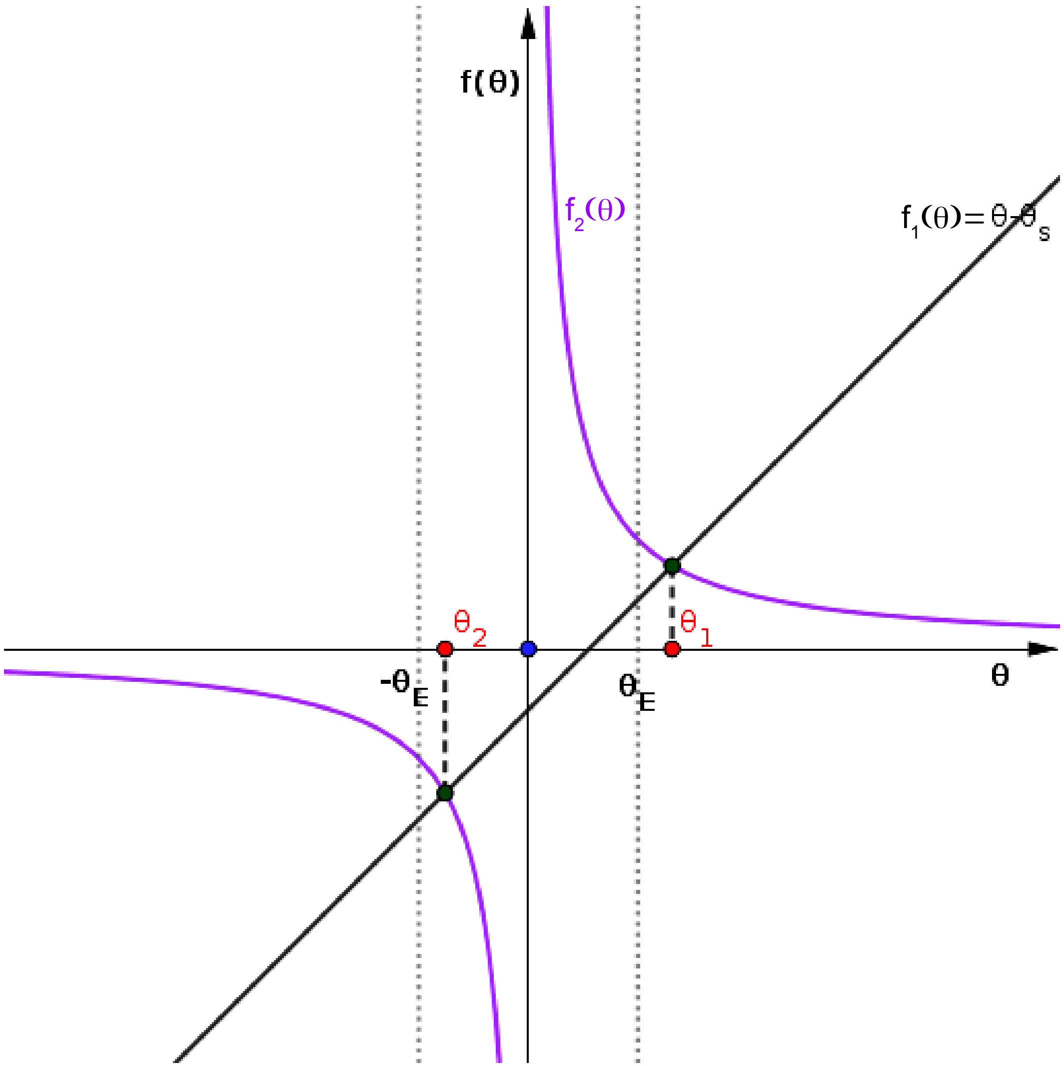
$$\theta_S = 0$$



Point-Mass Lens

$$\boxed{\theta - \theta_S = \frac{\theta_E^2}{\theta}}$$

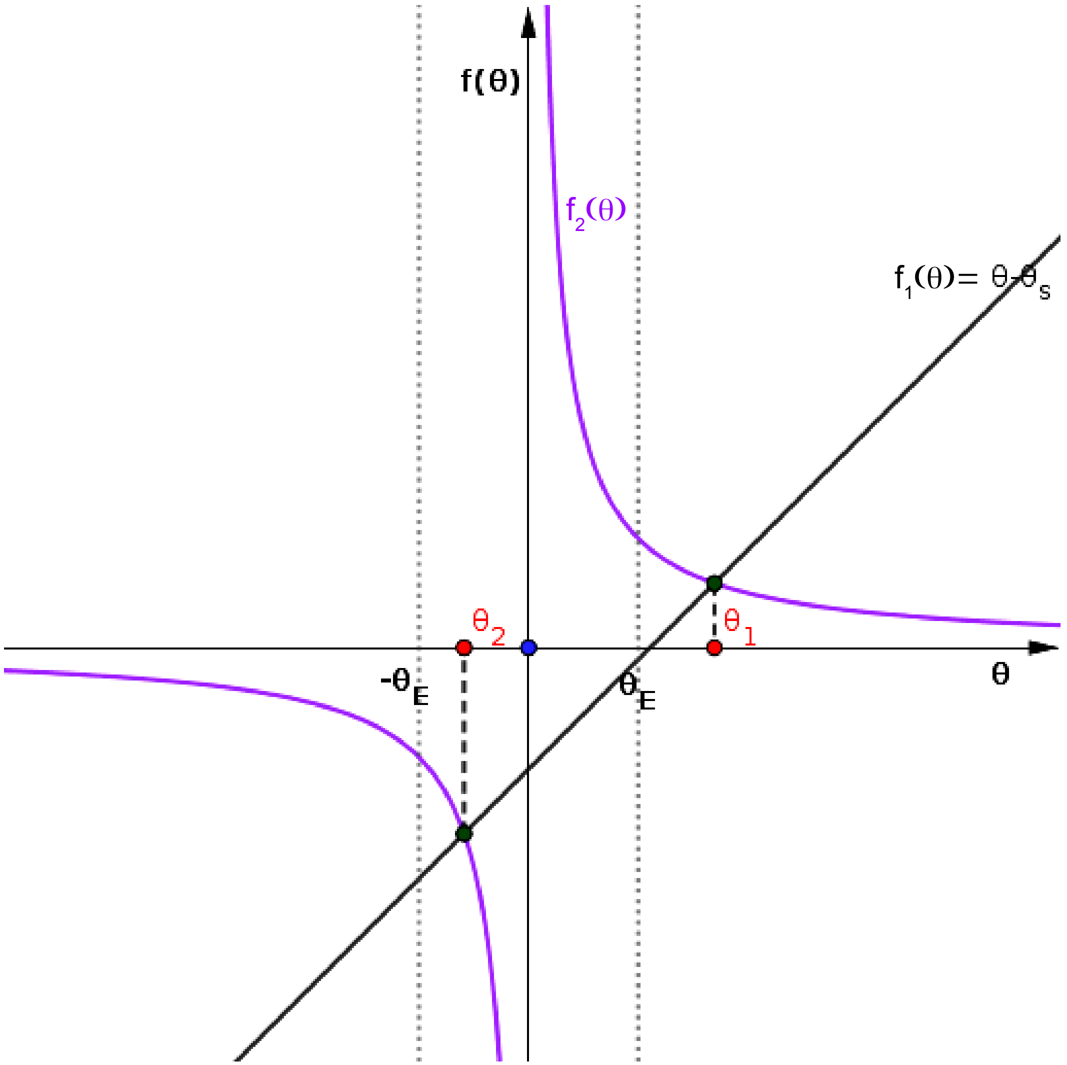
$\theta_S > 0$



Point-Mass Lens

$$\boxed{\theta - \theta_S = \frac{\theta_E^2}{\theta}}$$

$$\theta_S > 0$$



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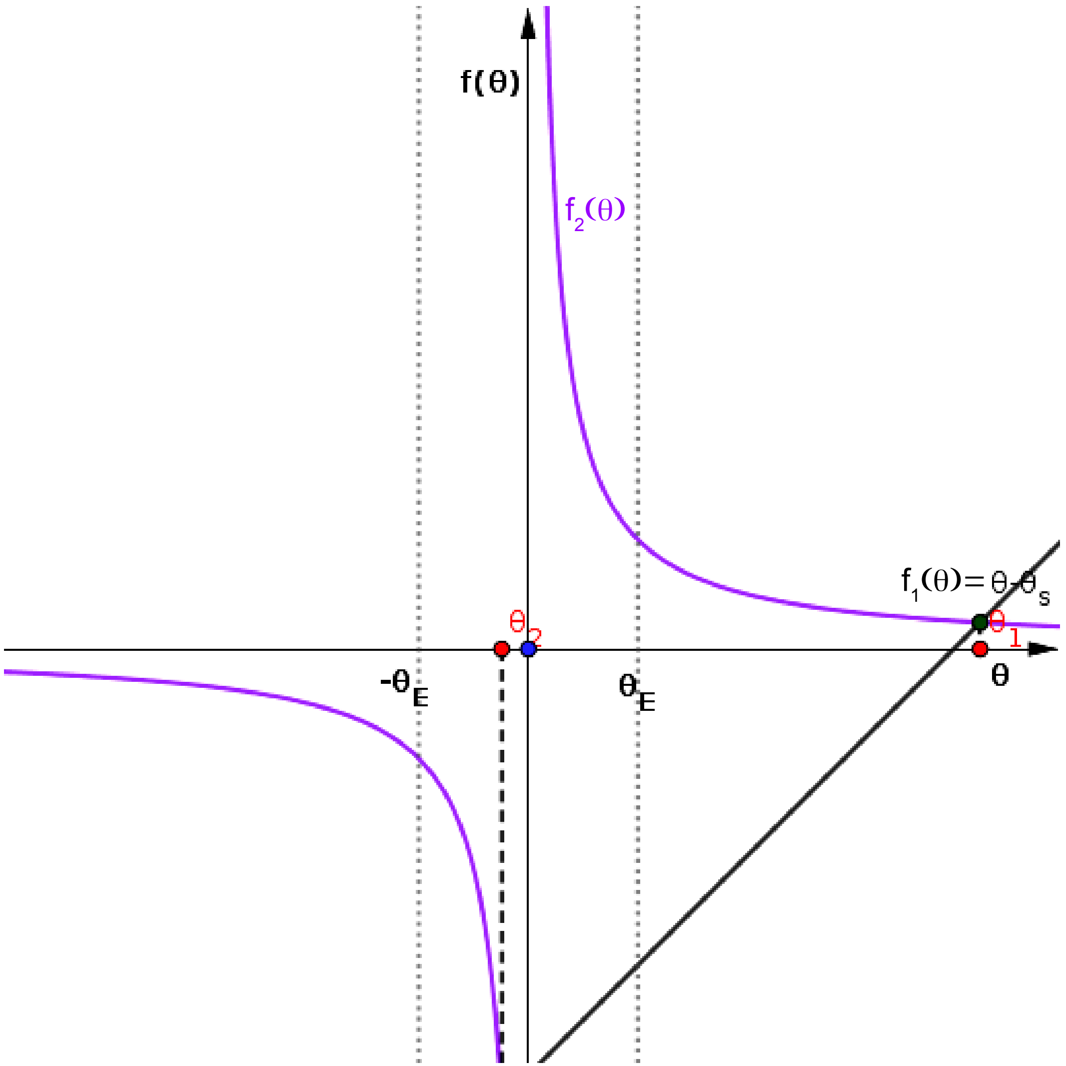
Point-Mass Lens

$$\boxed{\theta - \theta_S = \frac{\theta_E^2}{\theta}}$$

$$\theta_S > 0$$

- Or...

$$\boxed{\theta^2 - \theta_S^2 - \theta_E^2 = 0}$$



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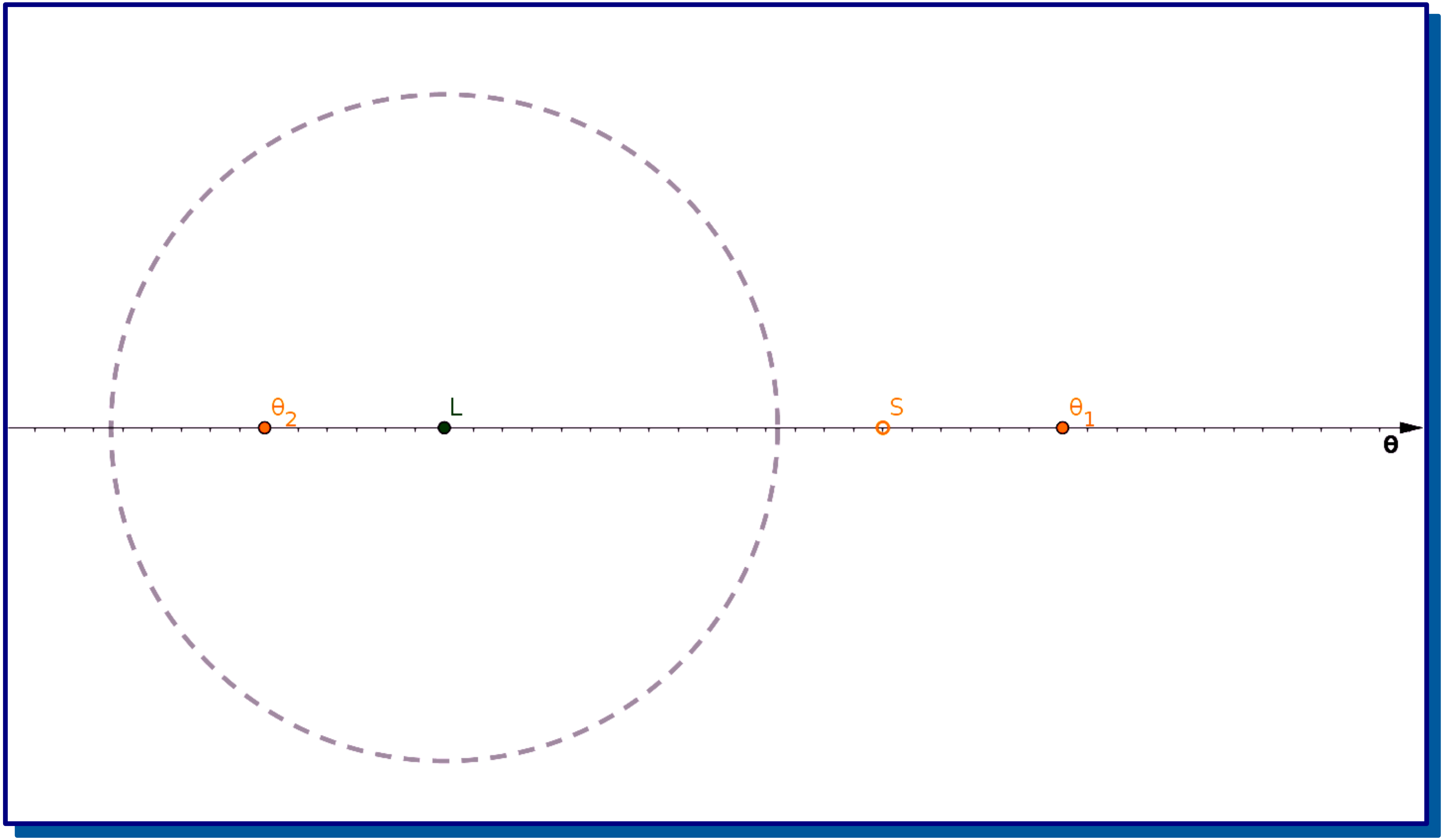

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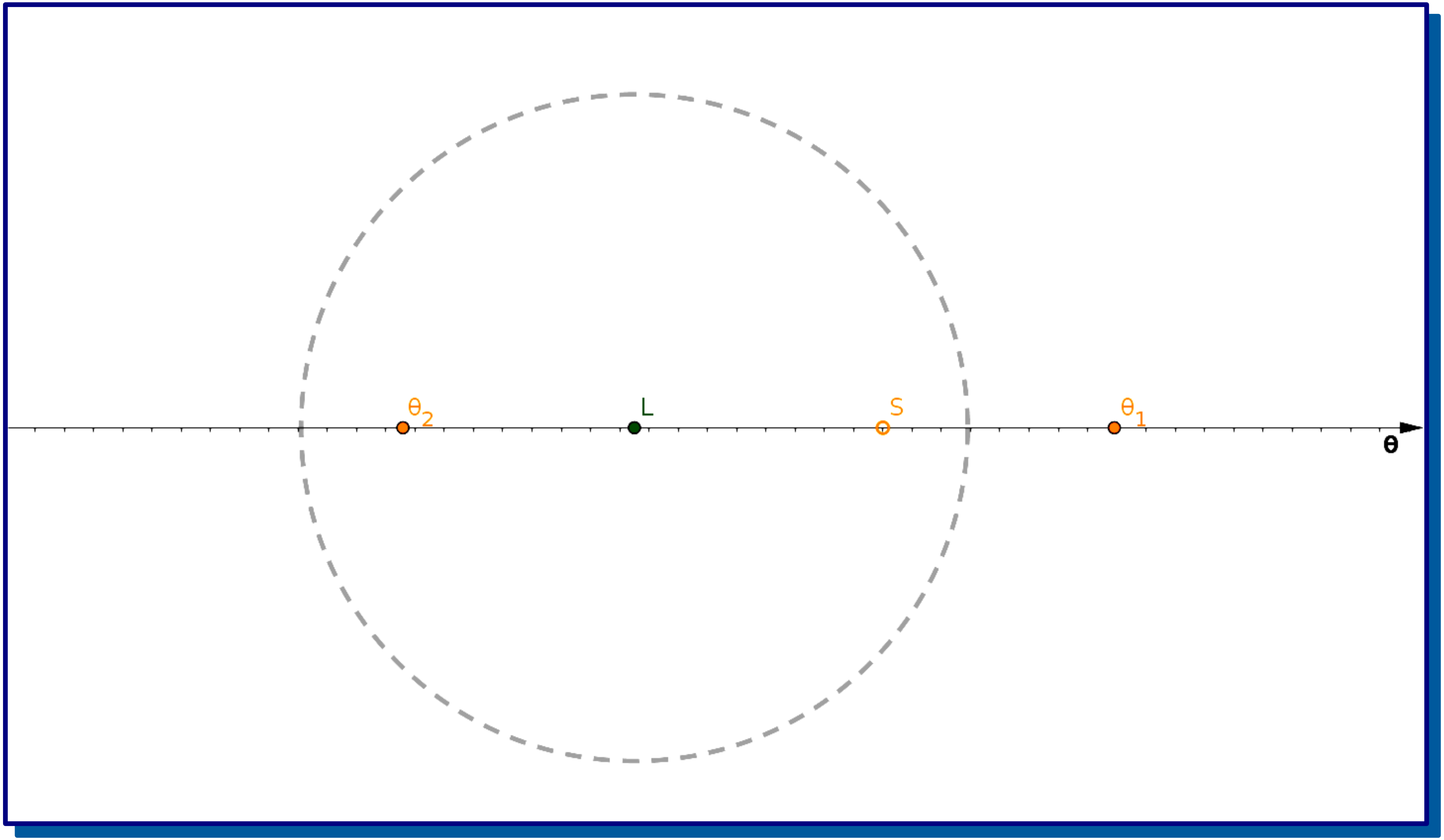

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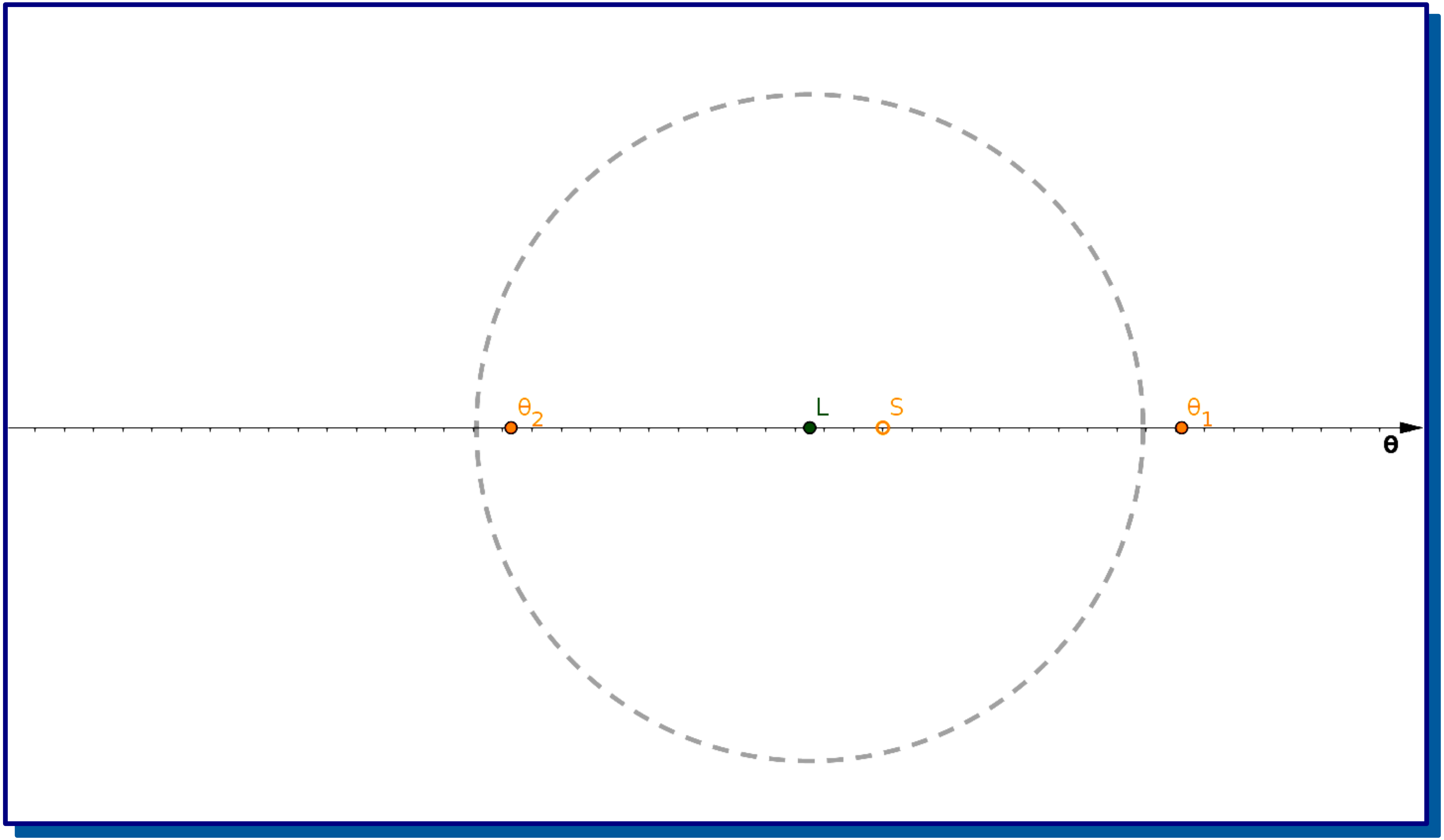

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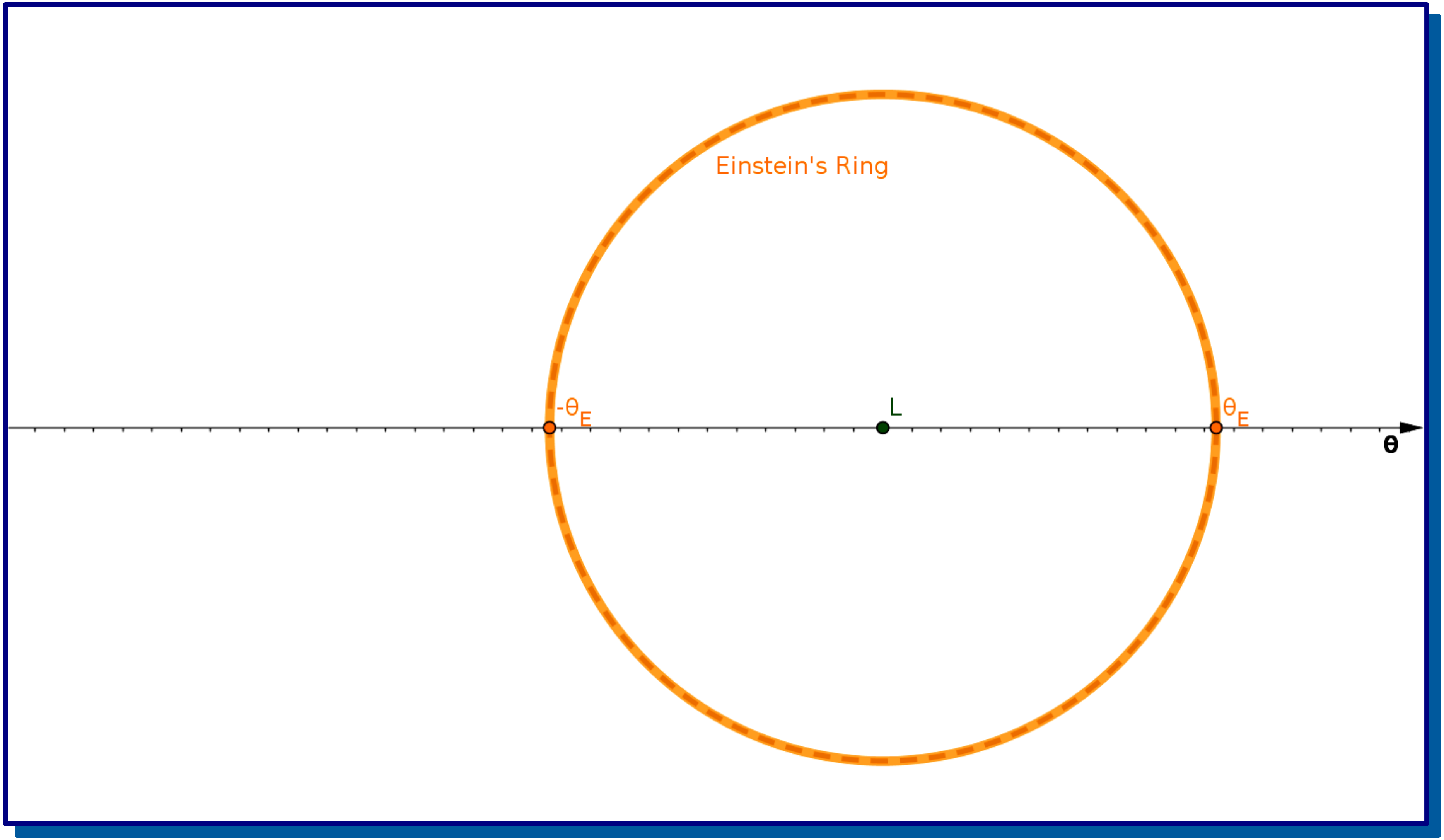

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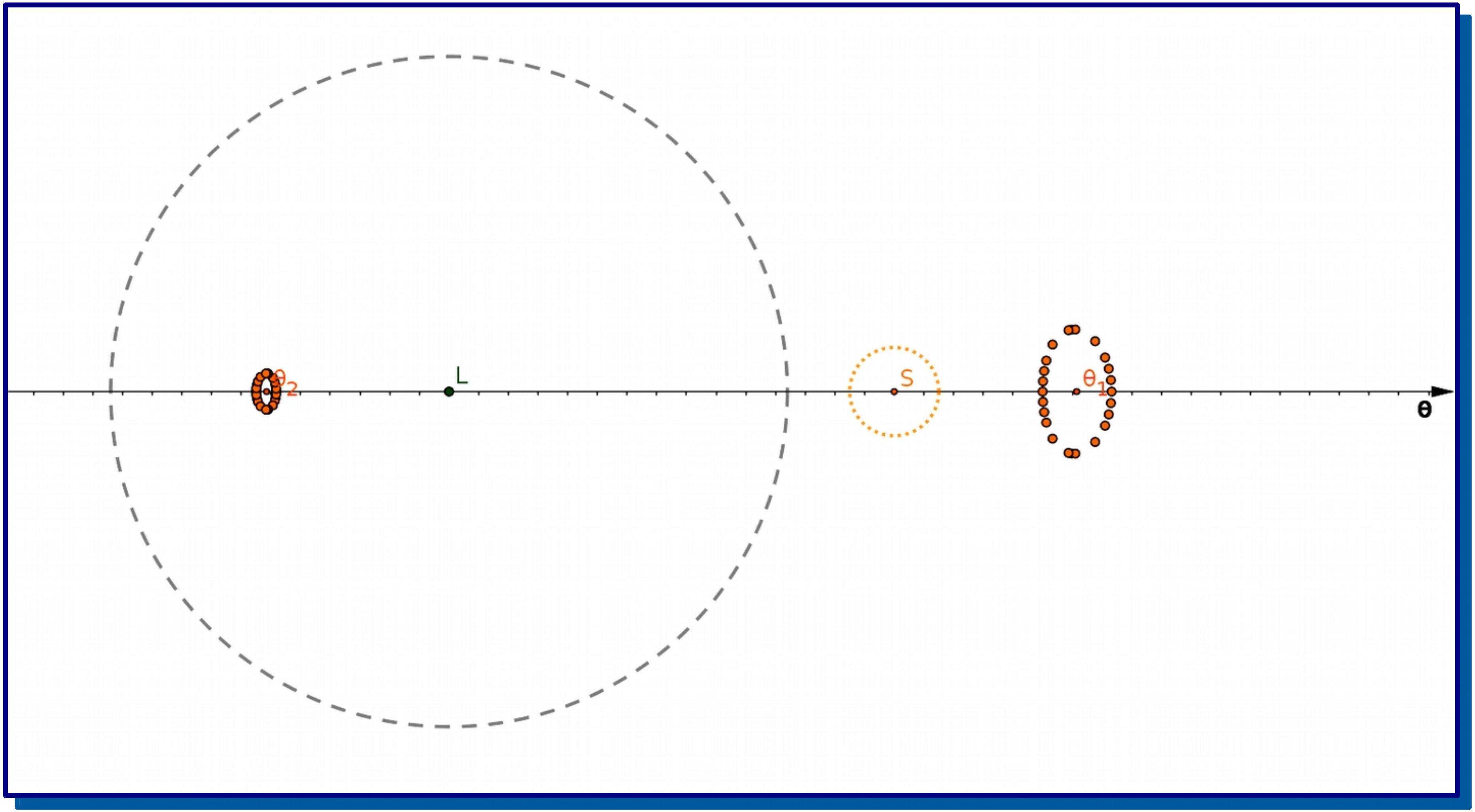
Extended Source



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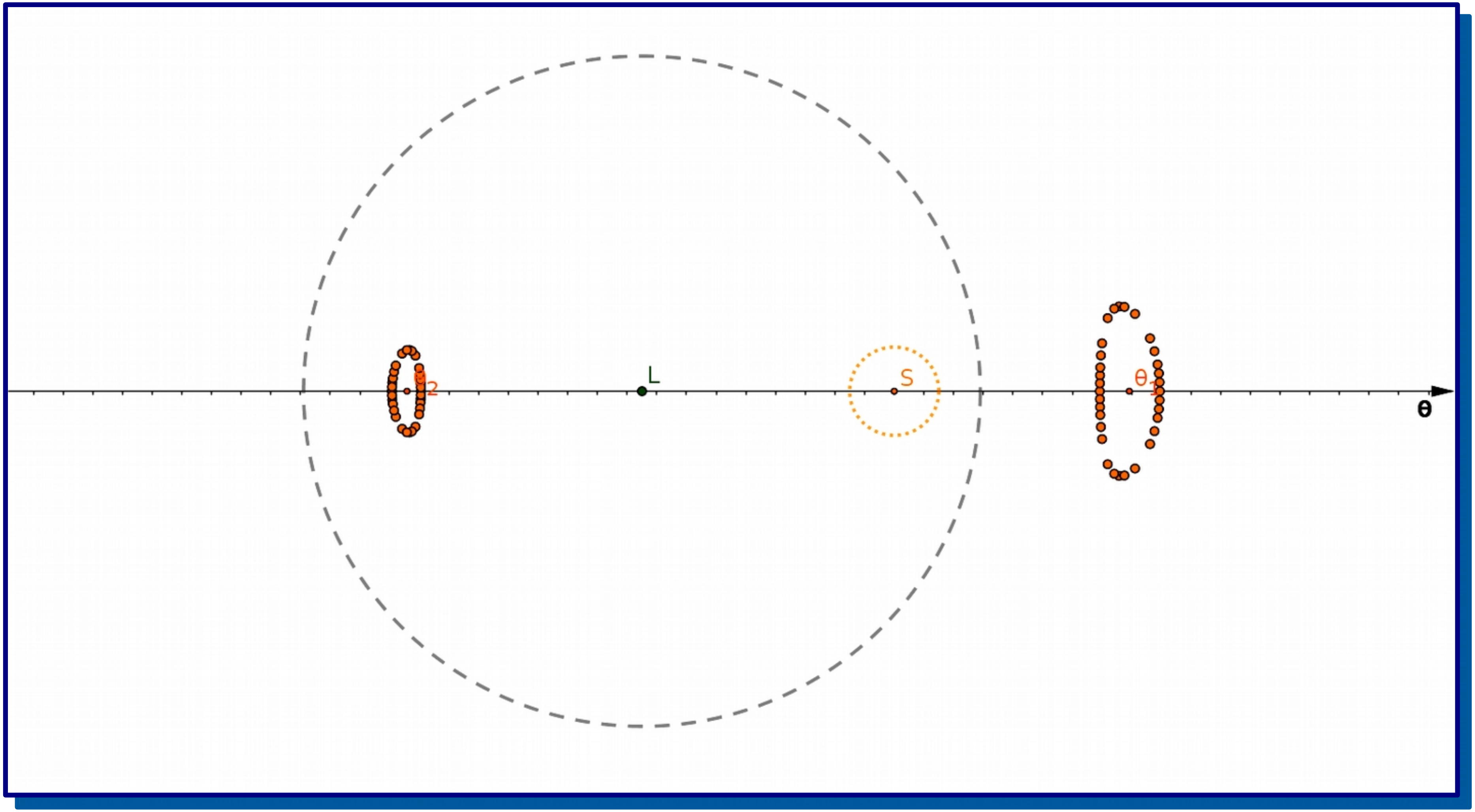

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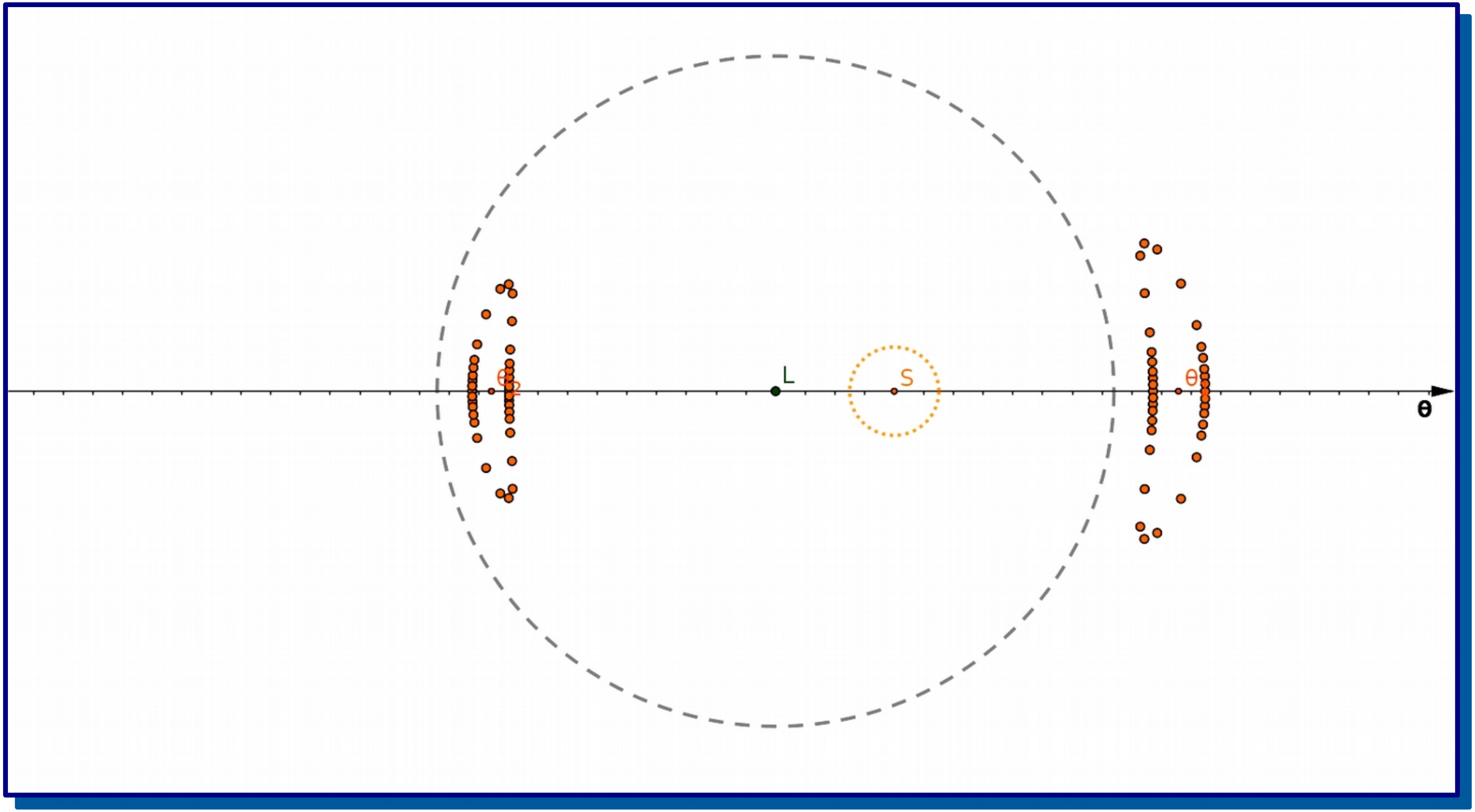

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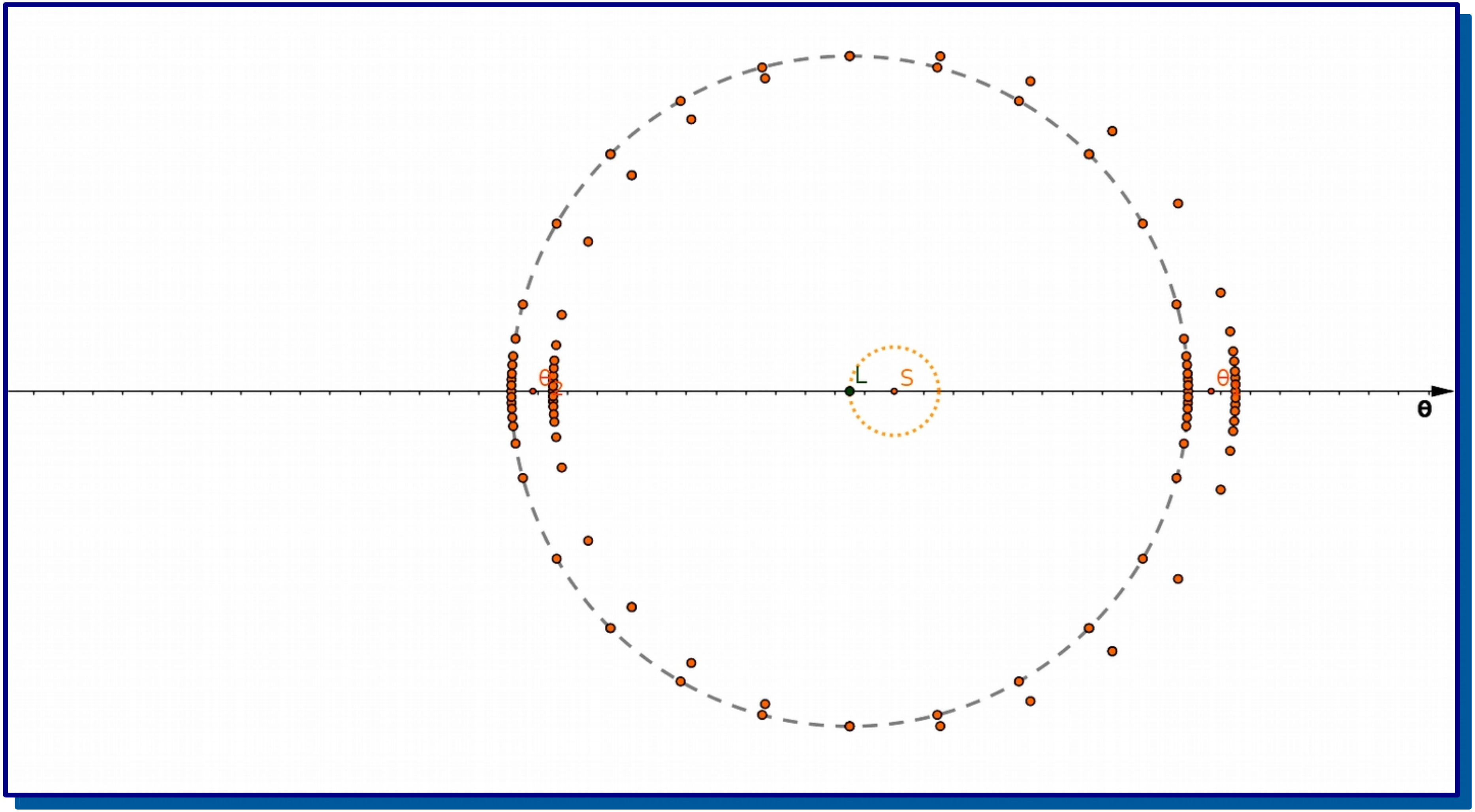

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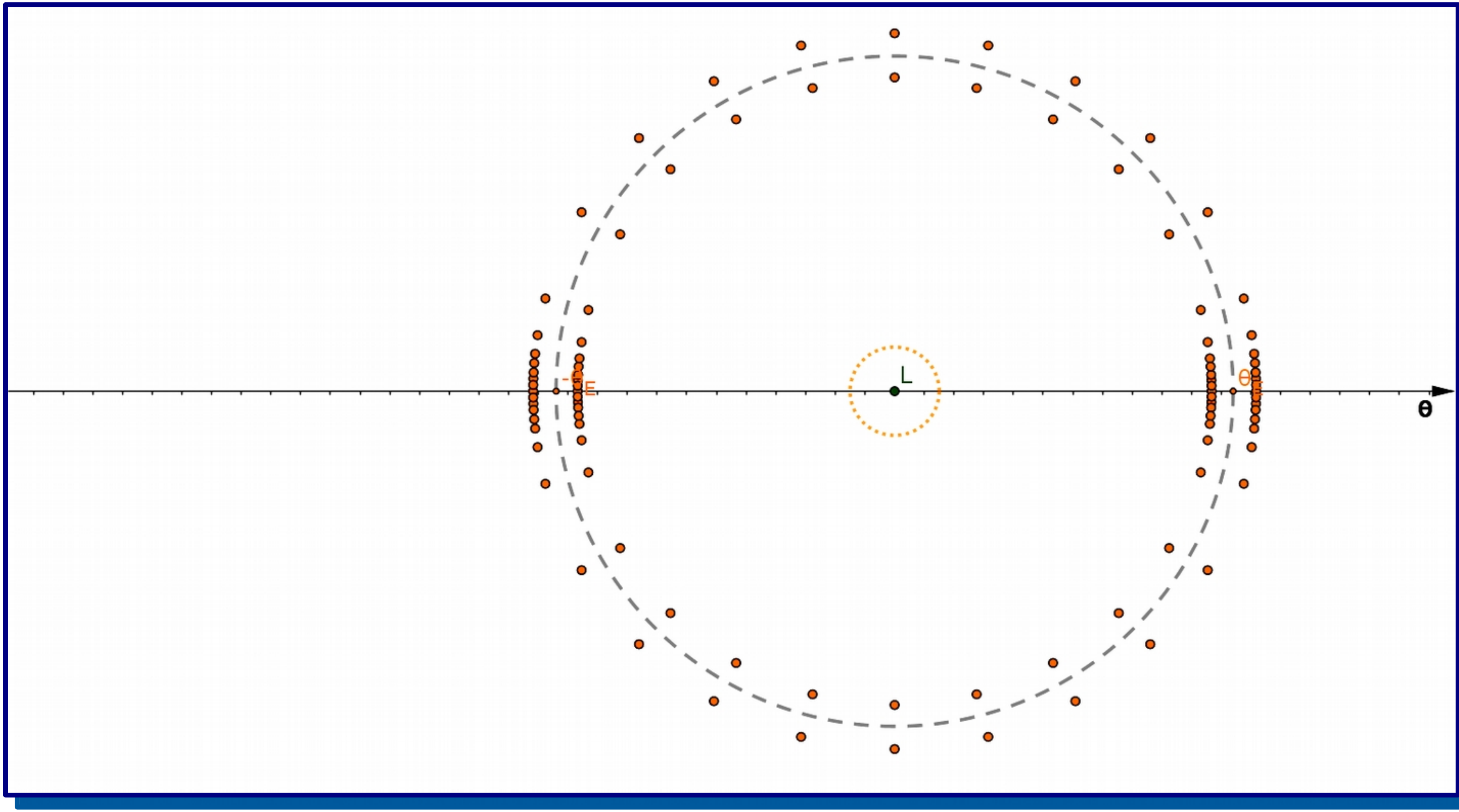
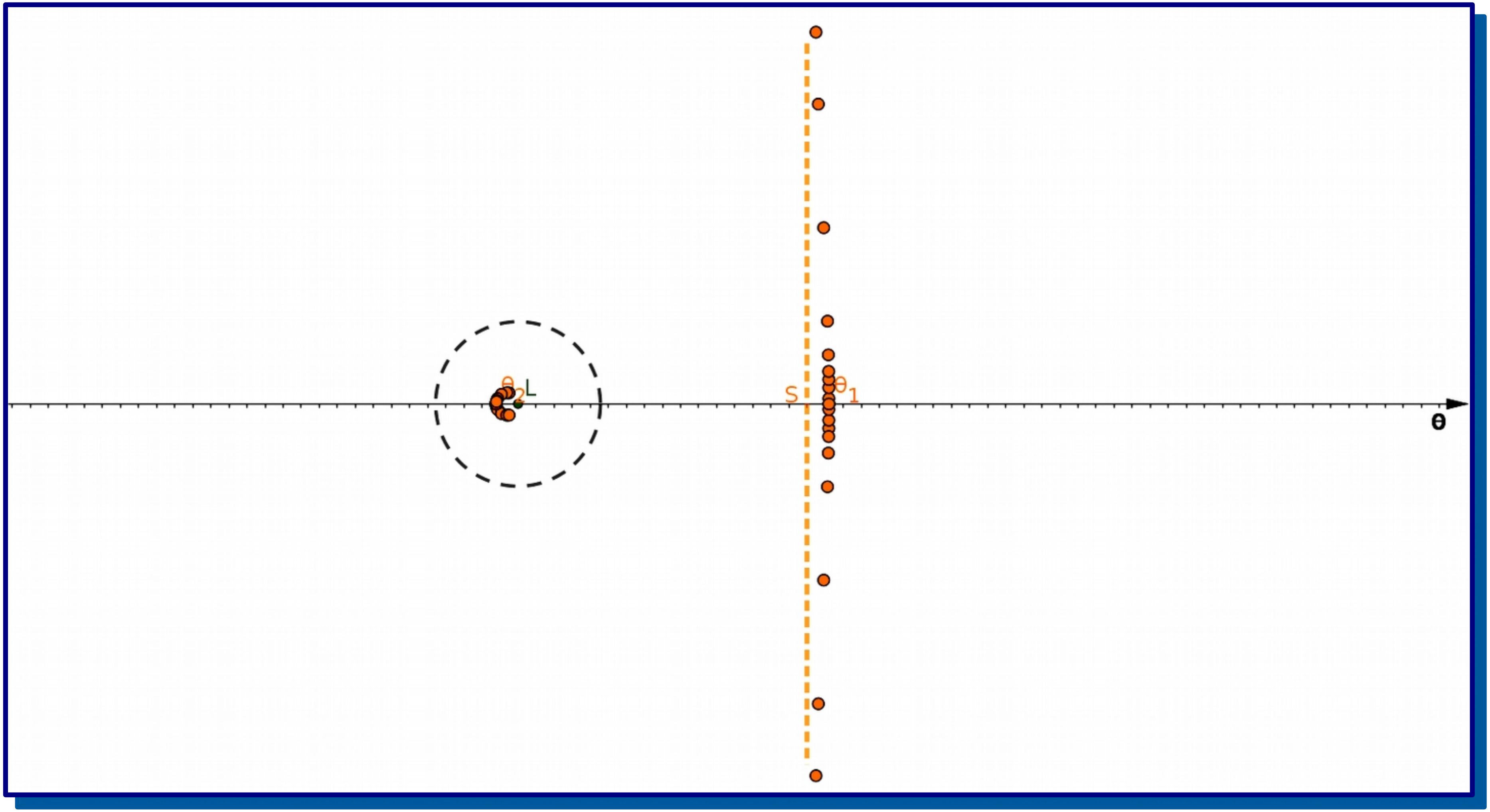


Line Source



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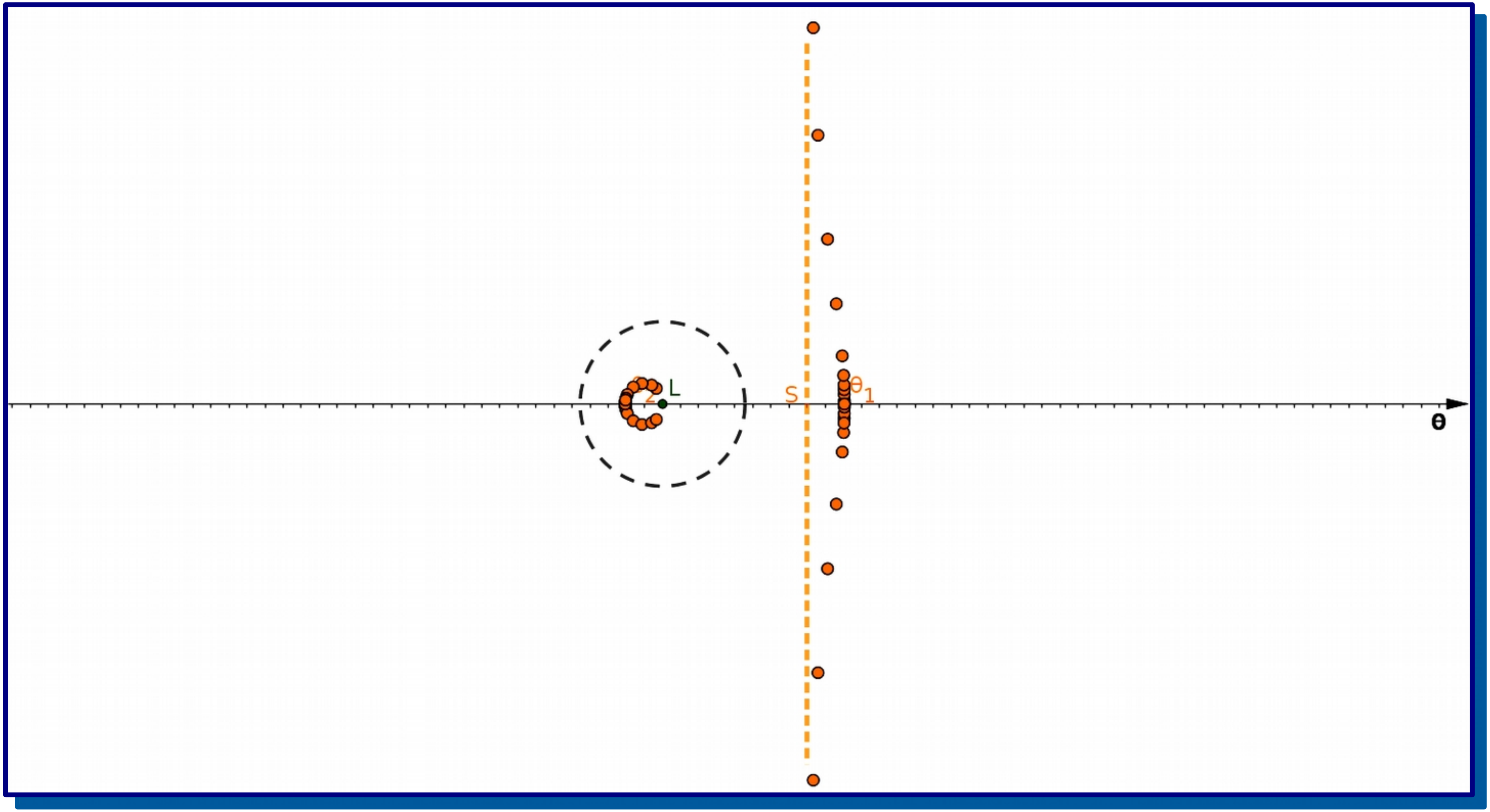
Line Source



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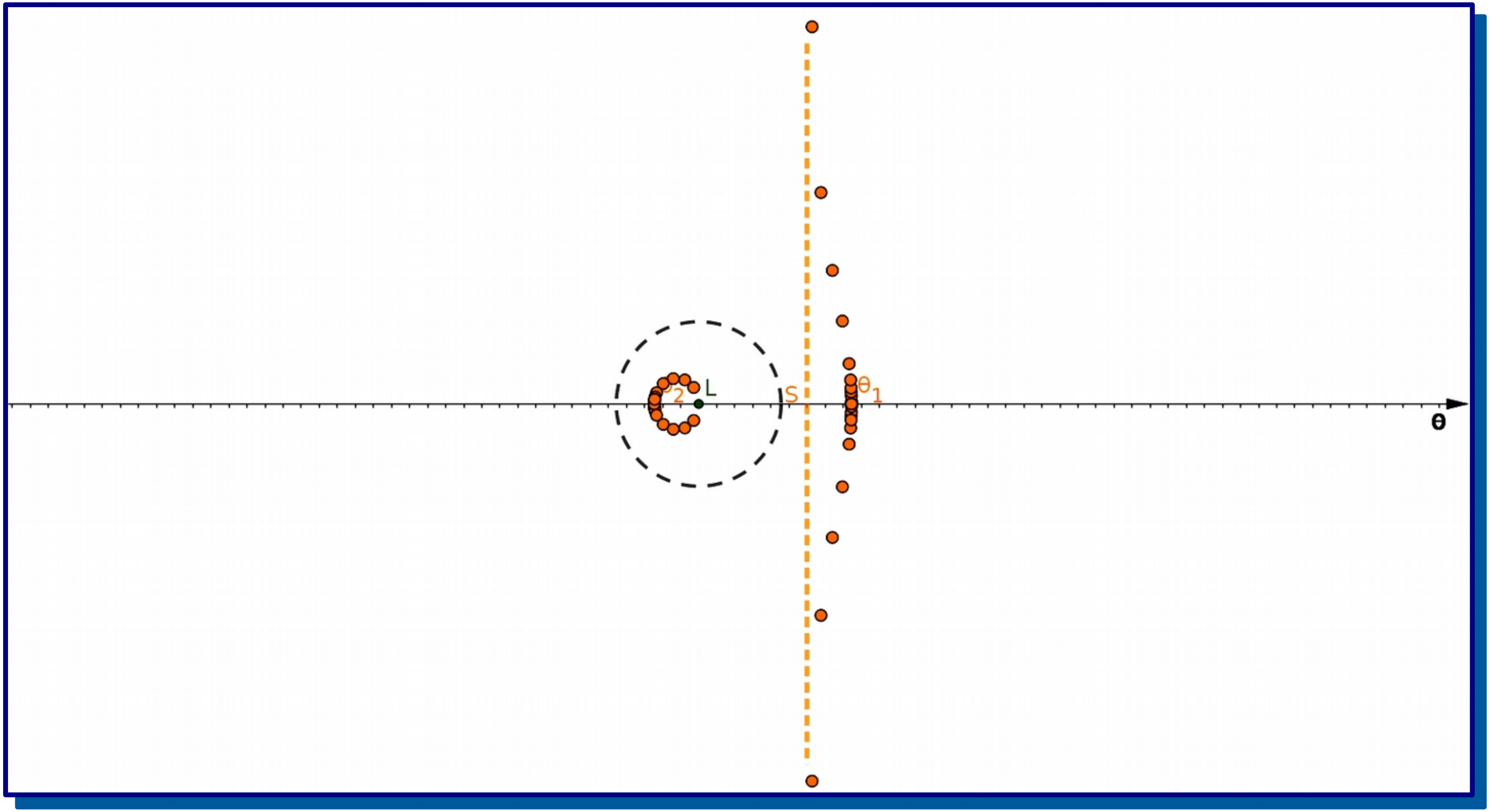

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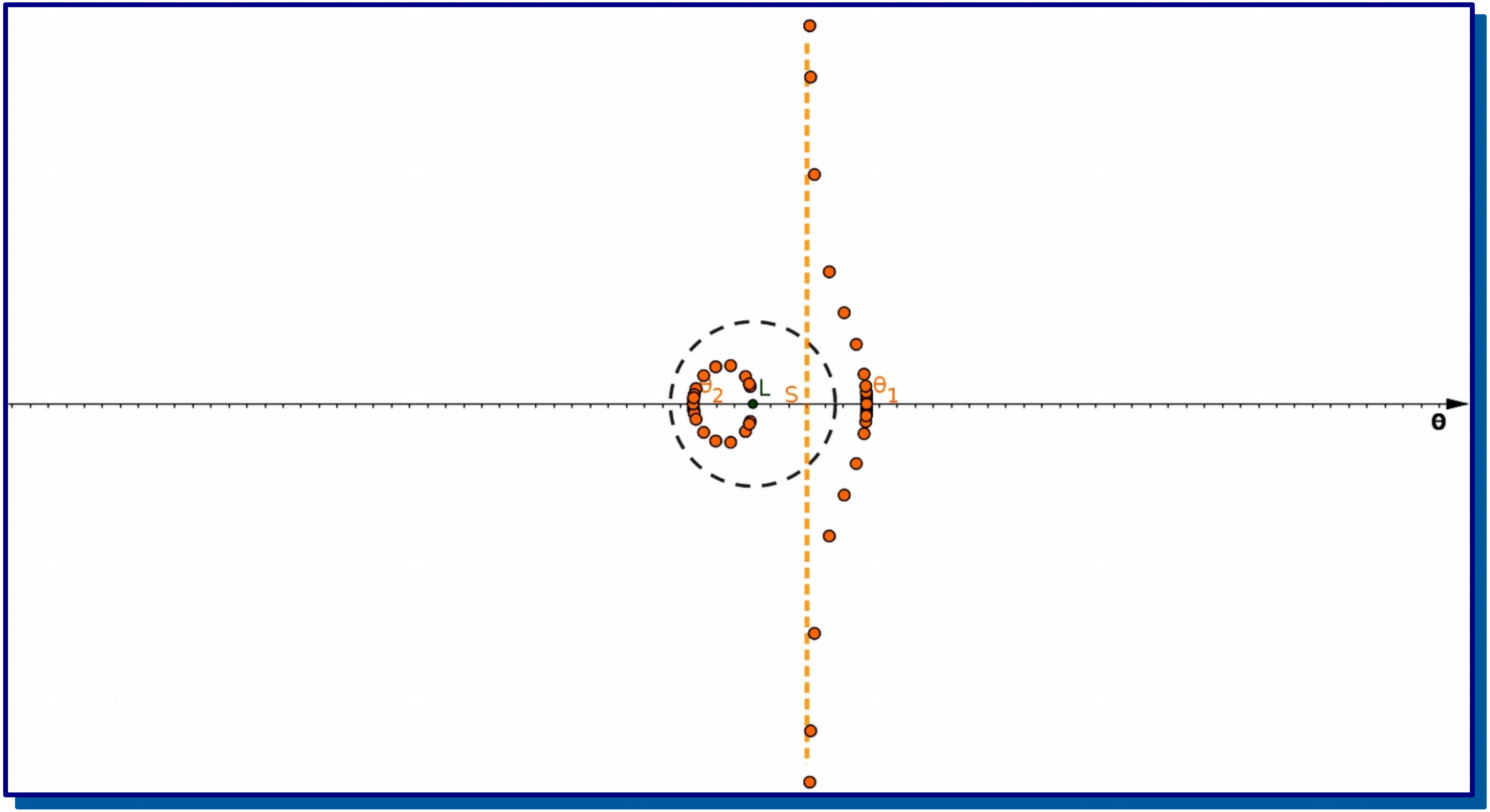

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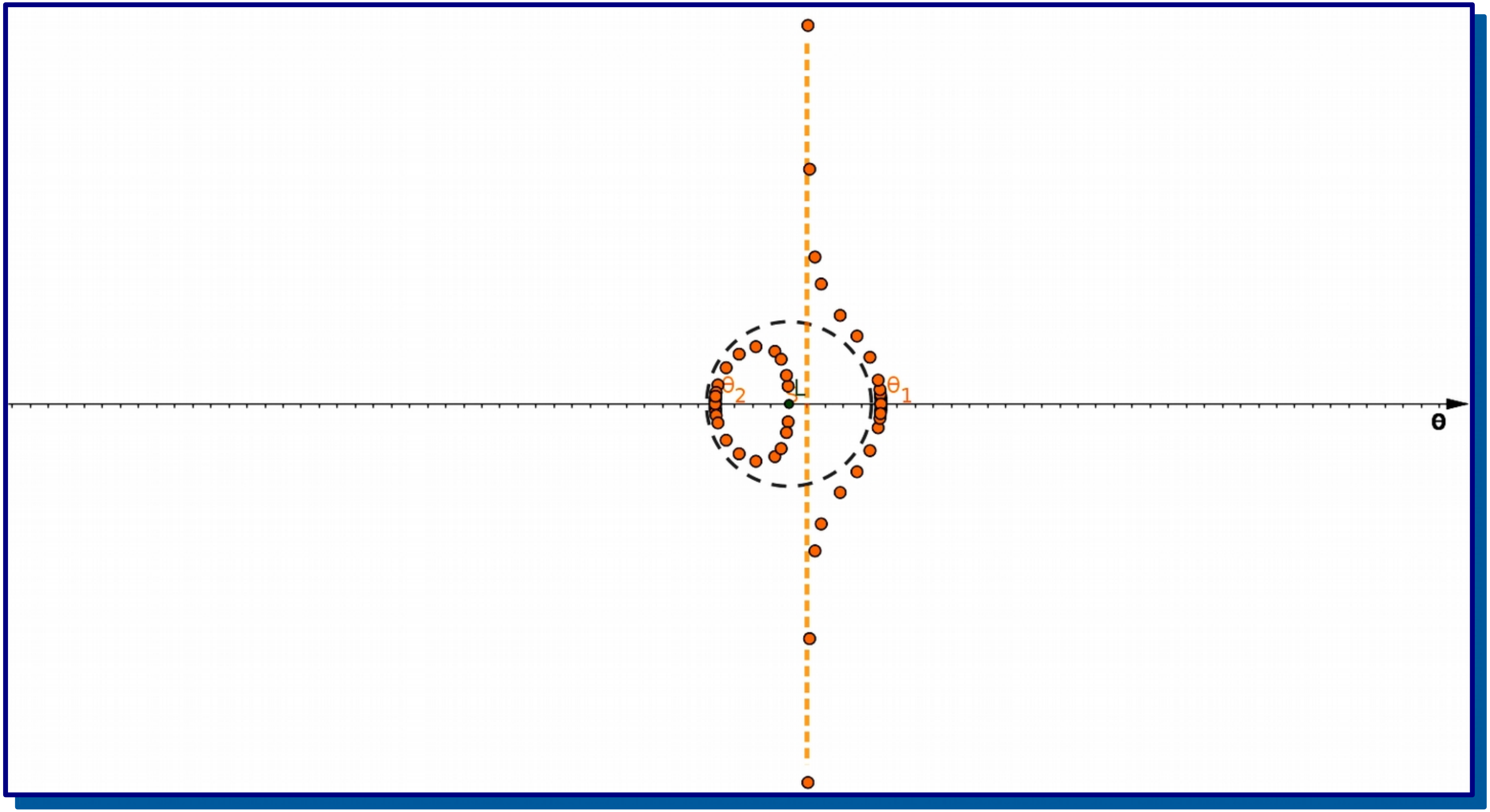

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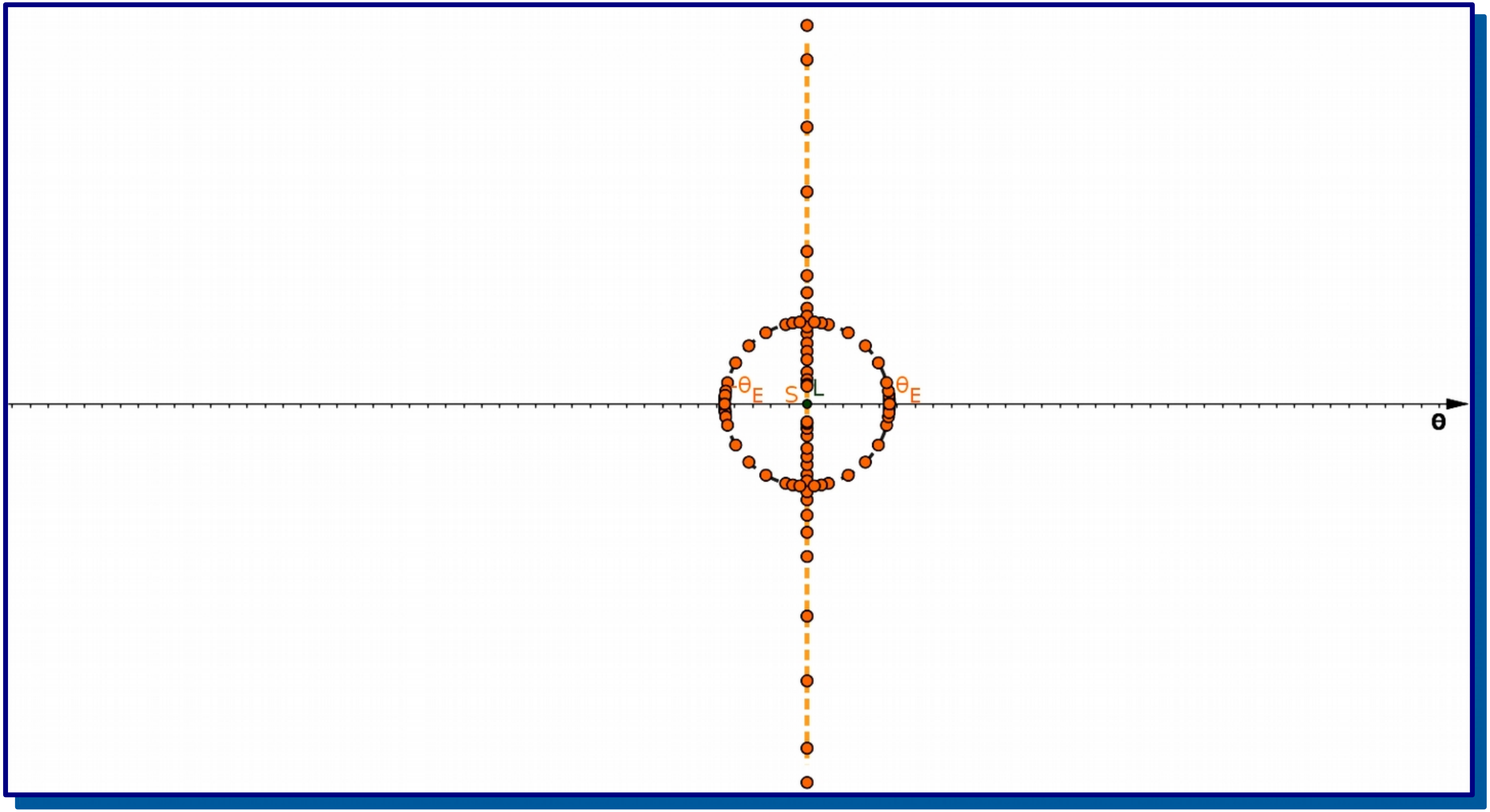
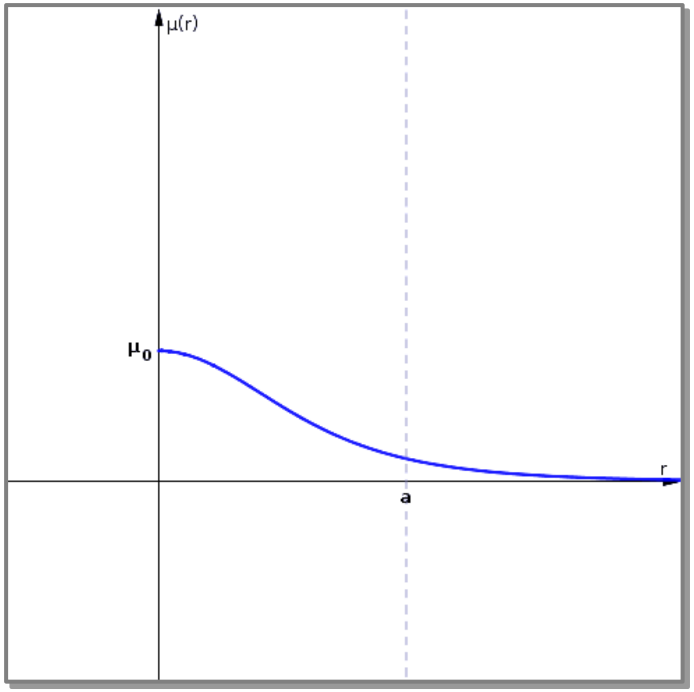


Plummer Sphere Lens

- Mass Density

$$\boxed{\mu(r) = \frac{3a^2}{4\pi} \frac{M}{(r^2+a^2)^{5/2}}}$$



- Mass:

$$M(r) = M \frac{r^3}{(r^2+a^2)^{3/2}} \longrightarrow \boxed{M(\rho) = M \frac{\rho^2}{(\rho^2+a^2)}}$$

Projected Mass

Plummer Sphere Lens

$$\theta-\theta_S=\frac{4G}{c^2}\frac{M(\theta)}{\theta}\frac{D_{LS}}{D_L D_S}=\frac{4G}{c^2}\frac{M\cdot\theta}{(\theta^2+\theta_a^2)}\frac{D_{LS}}{D_L D_S}$$

$$\theta_E=\frac{4G}{c^2}\frac{M(\theta_E)}{\theta_E}\frac{D_{LS}}{D_L D_S}=\frac{4G}{c^2}\frac{M\cdot\theta_E}{(\theta_E^2+\theta_a^2)}\frac{D_{LS}}{D_L D_S}$$

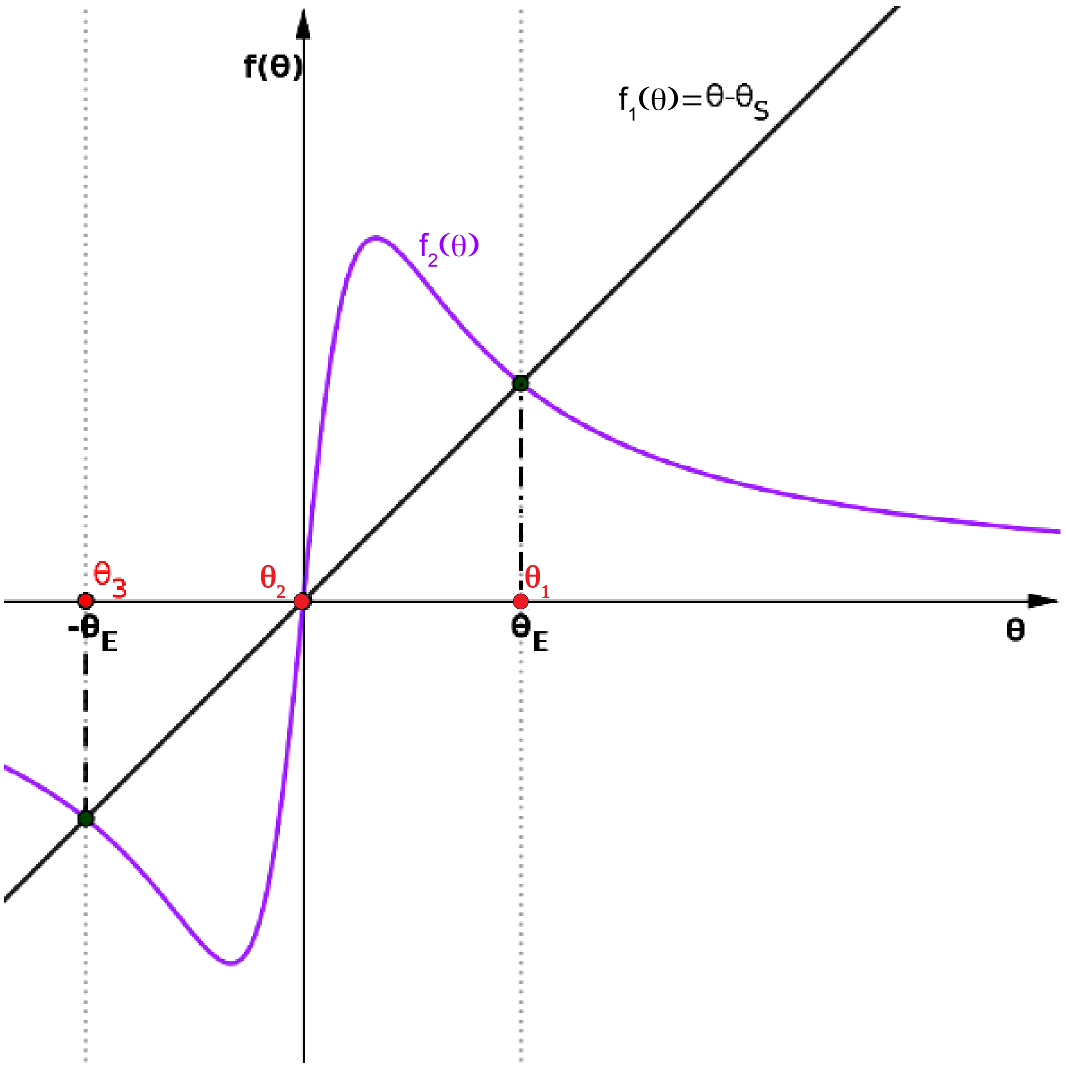
$$\boxed{\theta-\theta_S=\theta\frac{(\theta_E^2+\theta_a^2)}{(\theta^2+\theta_a^2)}}$$

Lens equation in terms of Einstein radius

Plummer Sphere Lens

$$\boxed{\theta-\theta_S=\theta\,\frac{\left(\theta_E^2+\theta_a^2\right)}{\left(\theta^2+\theta_a^2\right)}}$$

$$\theta_S=0$$



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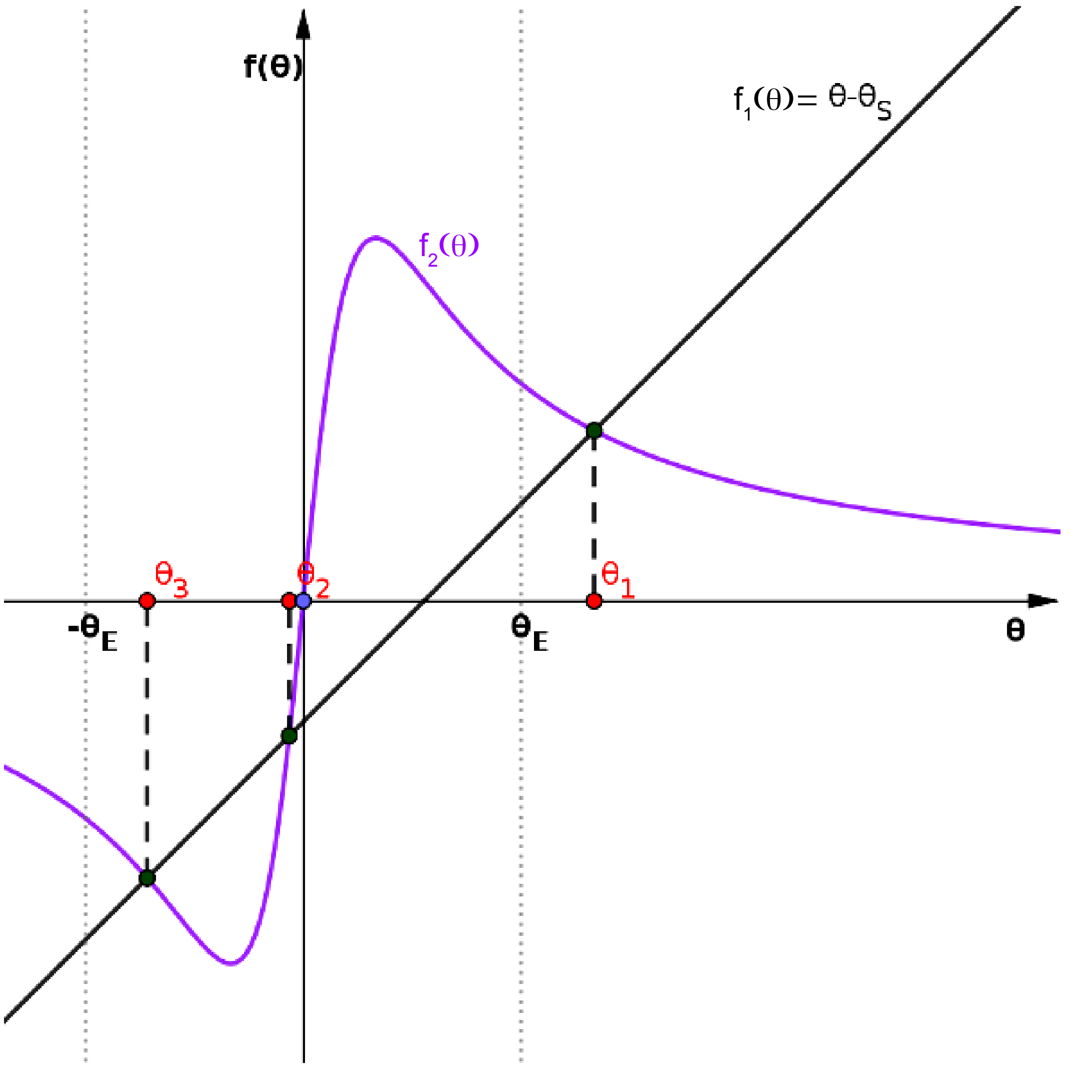
Plummer Sphere Lens

$$\boxed{\theta - \theta_S = \theta \frac{\left(\theta_E^2 + \theta_a^2\right)}{\left(\theta^2 + \theta_a^2\right)}}$$

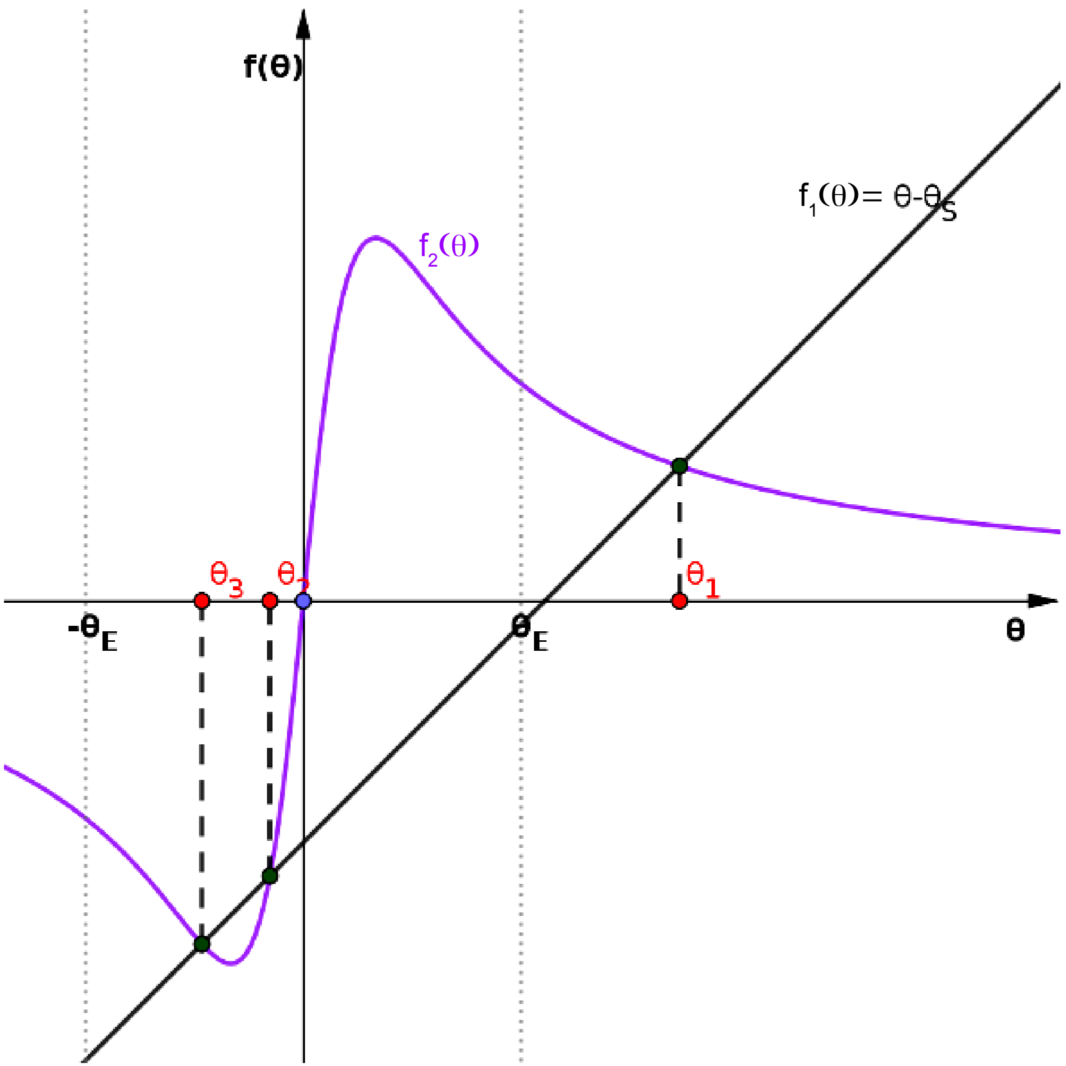
$\theta_S > 0$



Plummer Sphere Lens

$$\boxed{\theta - \theta_S = \theta \frac{\left(\theta_E^2 + \theta_a^2\right)}{\left(\theta^2 + \theta_a^2\right)}}$$

$$\theta_S > 0$$

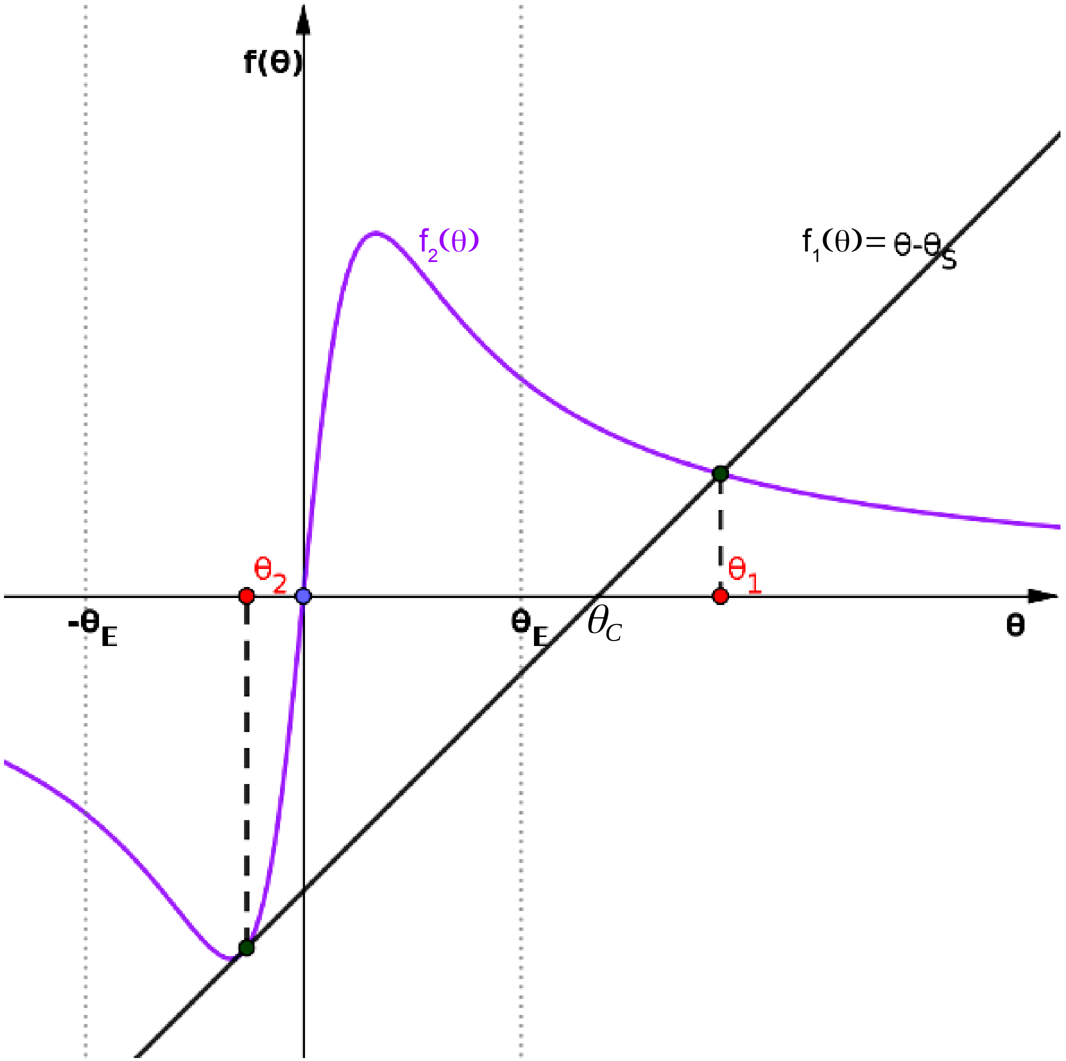


Plummer Sphere Lens

$$\boxed{\theta - \theta_S = \theta \frac{\left(\theta_E^2 + \theta_a^2\right)}{\left(\theta^2 + \theta_a^2\right)}}$$

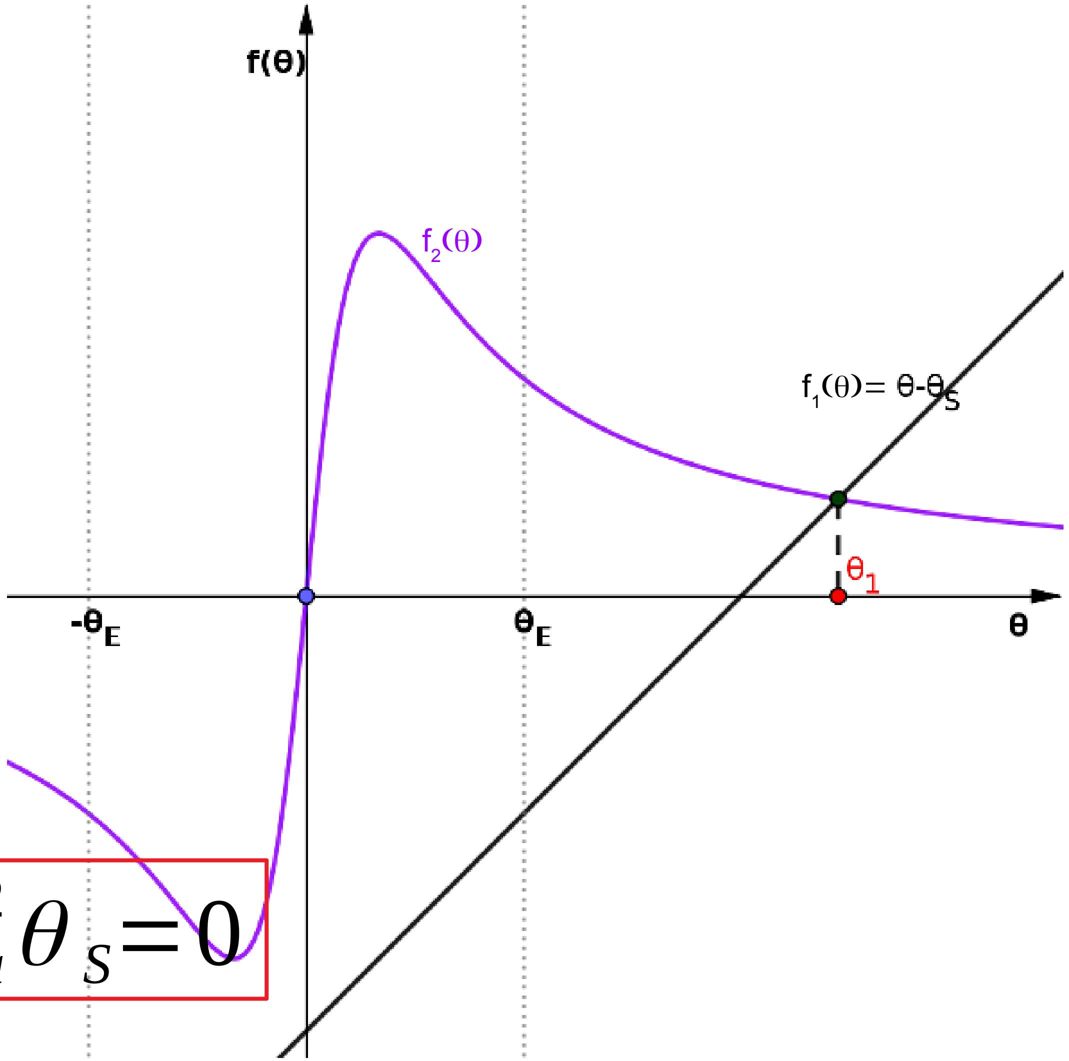
$\theta_S > 0$

$\theta_S = \theta_C$



Plummer Sphere Lens

$$\theta-\theta_S=\theta\frac{\left(\theta_E^2+\theta_a^2\right)}{\left(\theta^2+\theta_a^2\right)}$$

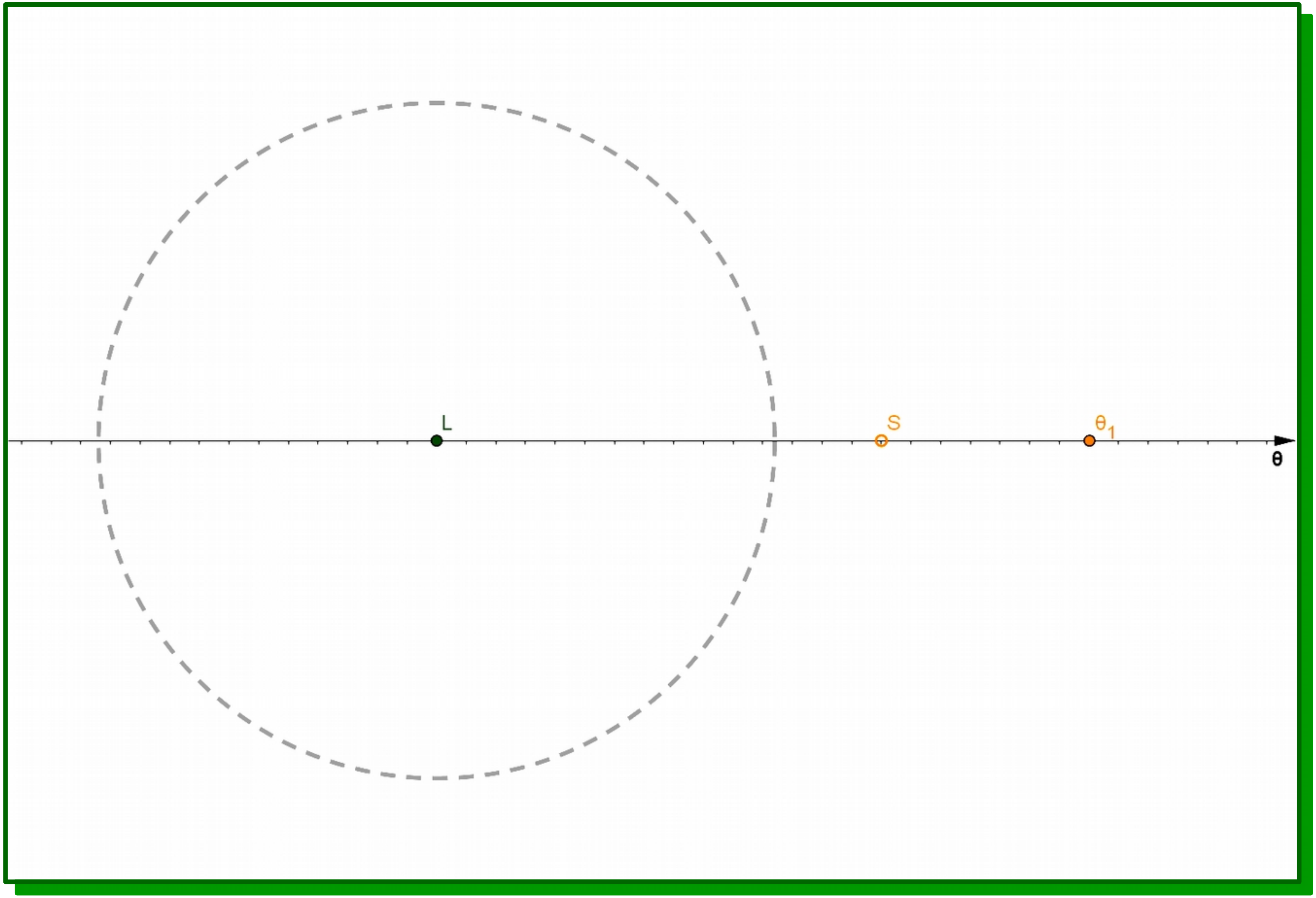


$\theta_S>0$

- Or...

$$\theta^3-\theta_S\theta^2-\theta_E^2\theta-\theta_a^2\theta_S=0$$

Point-Like Source



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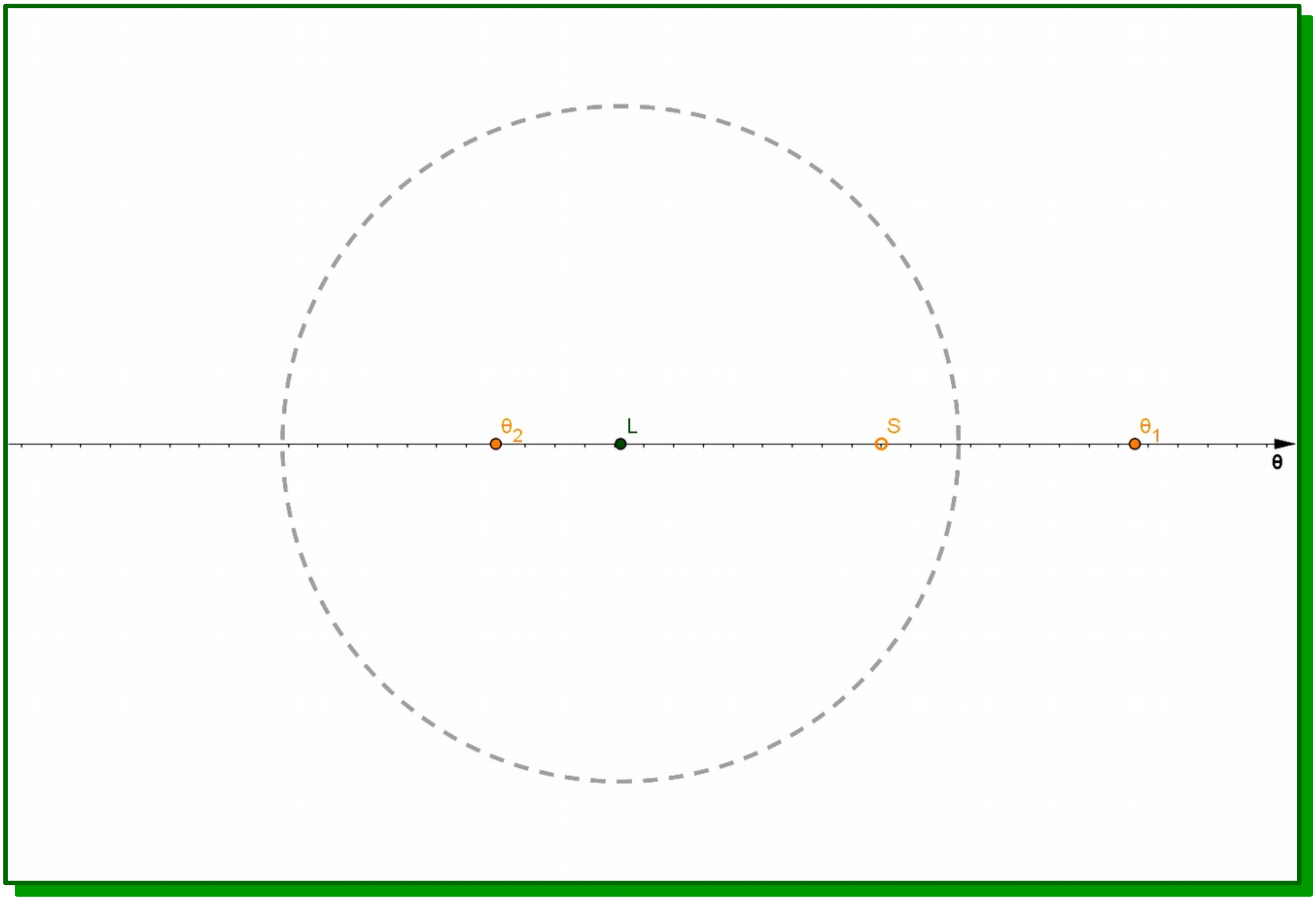

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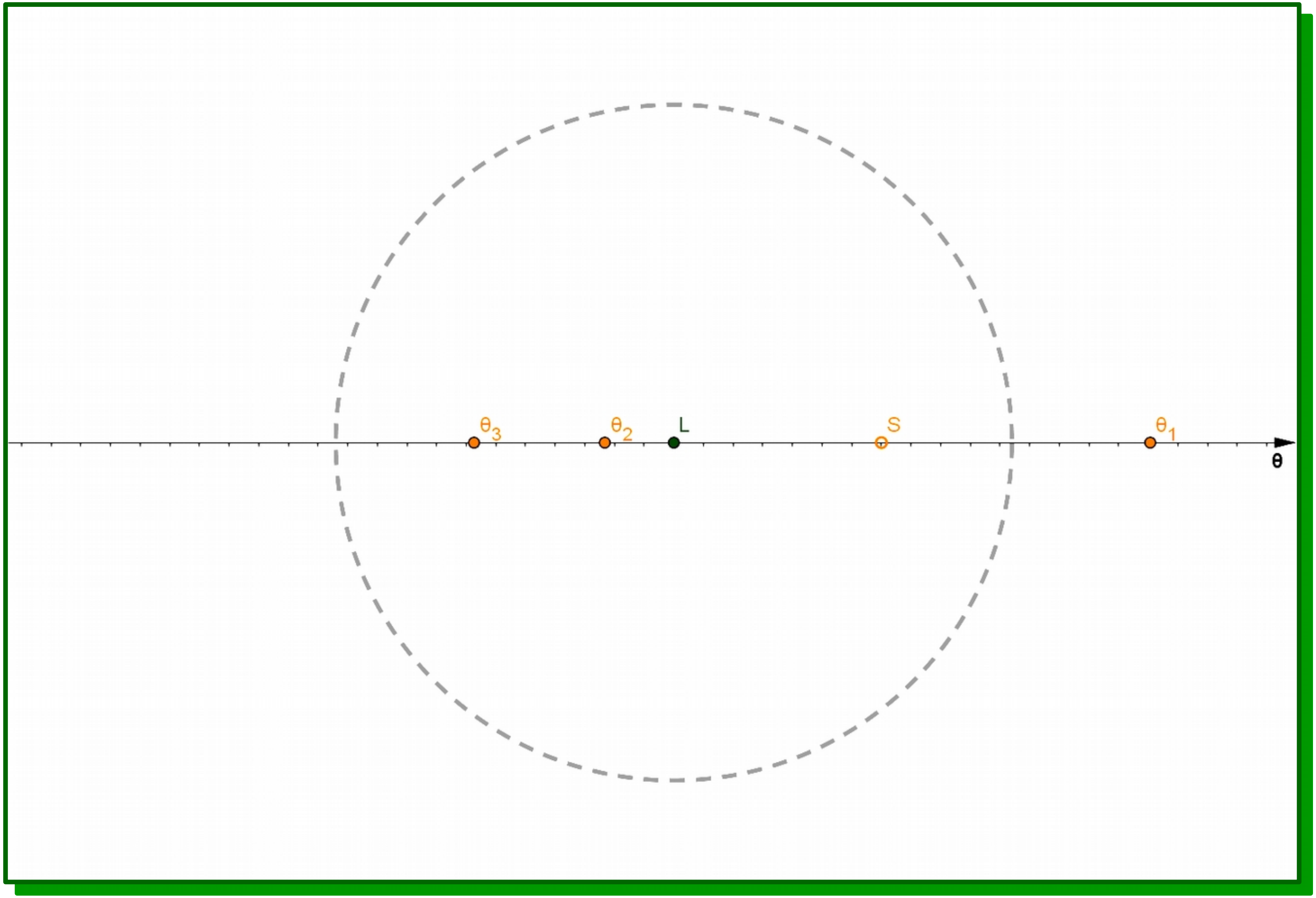

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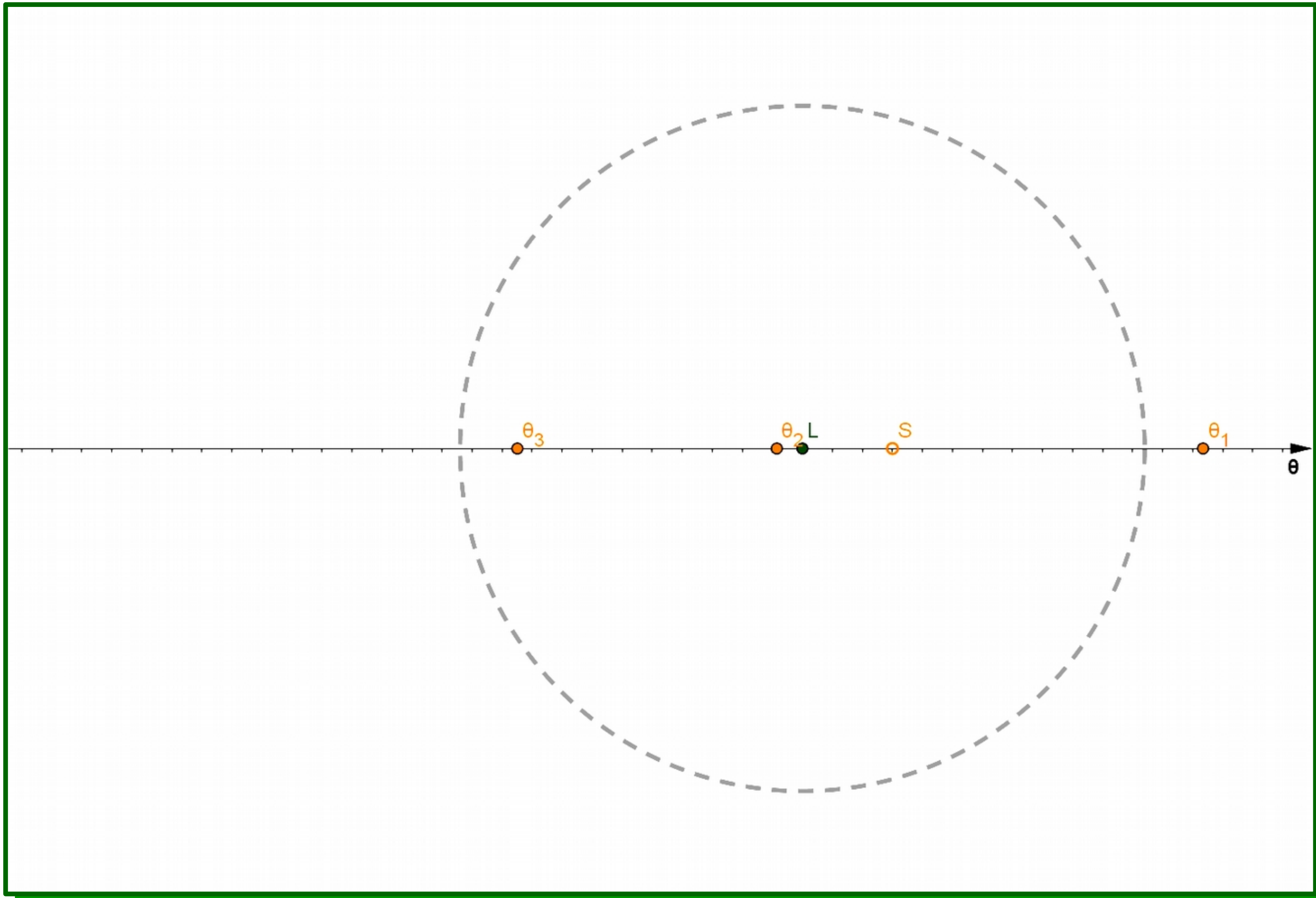

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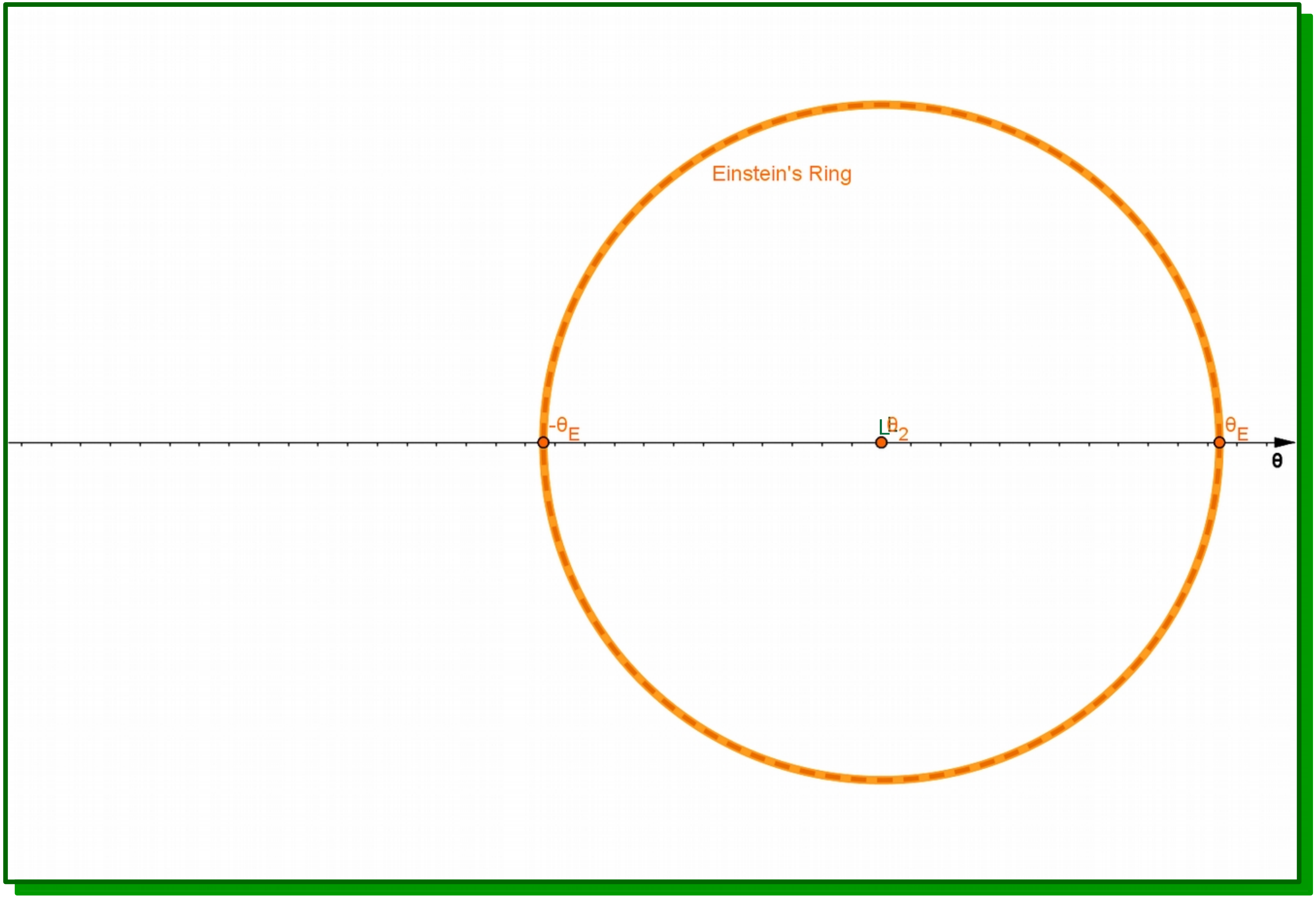

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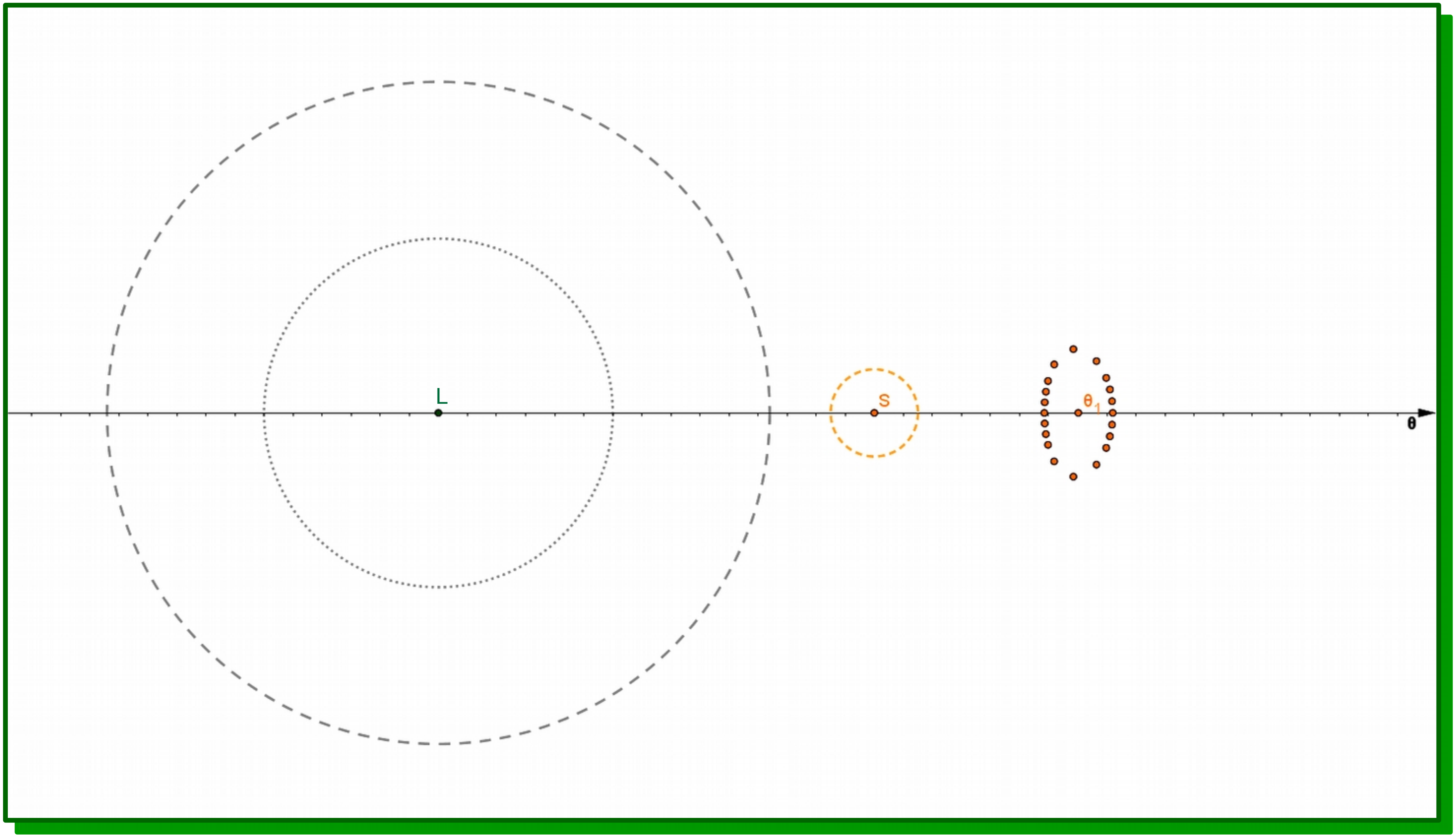

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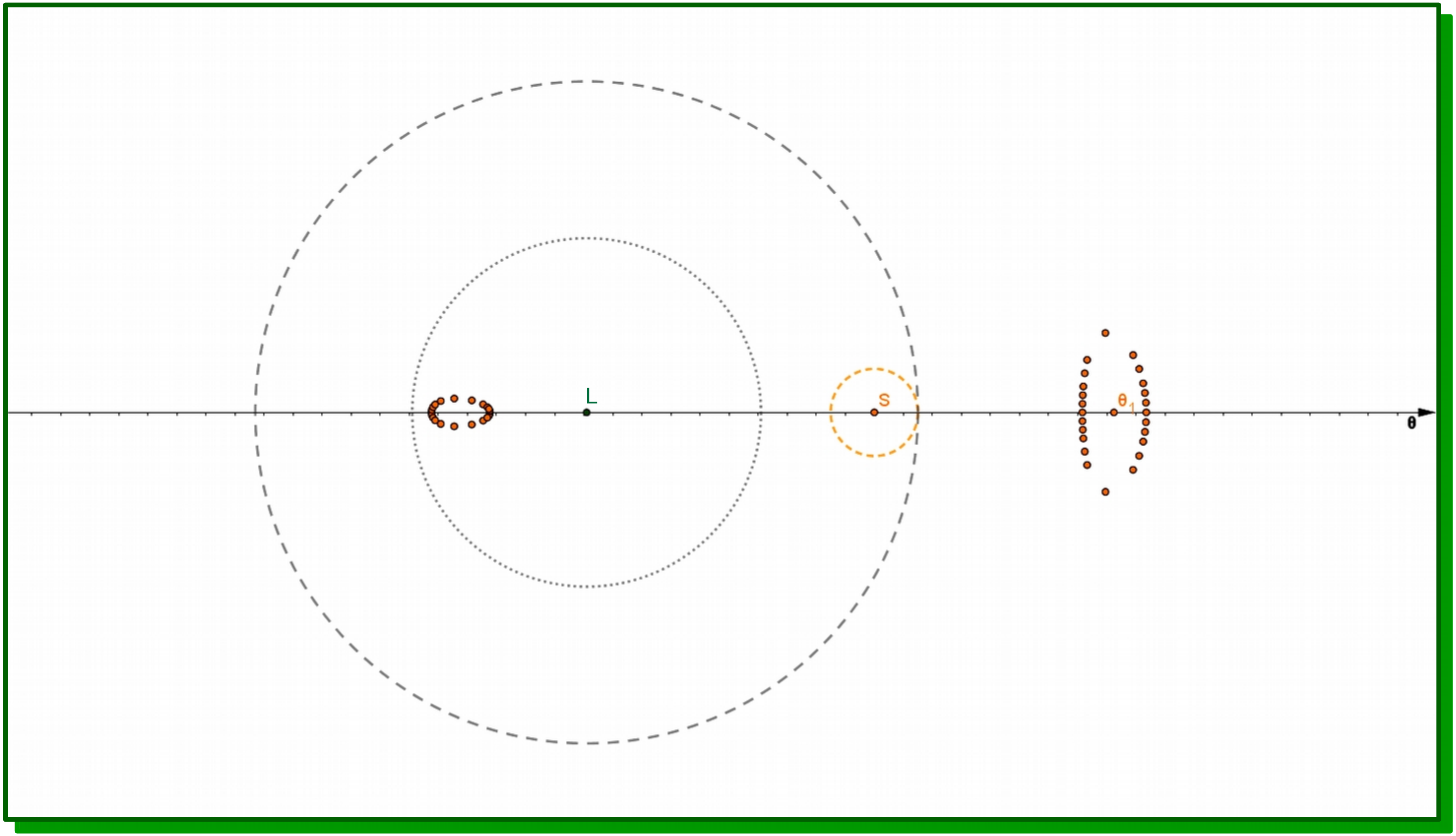

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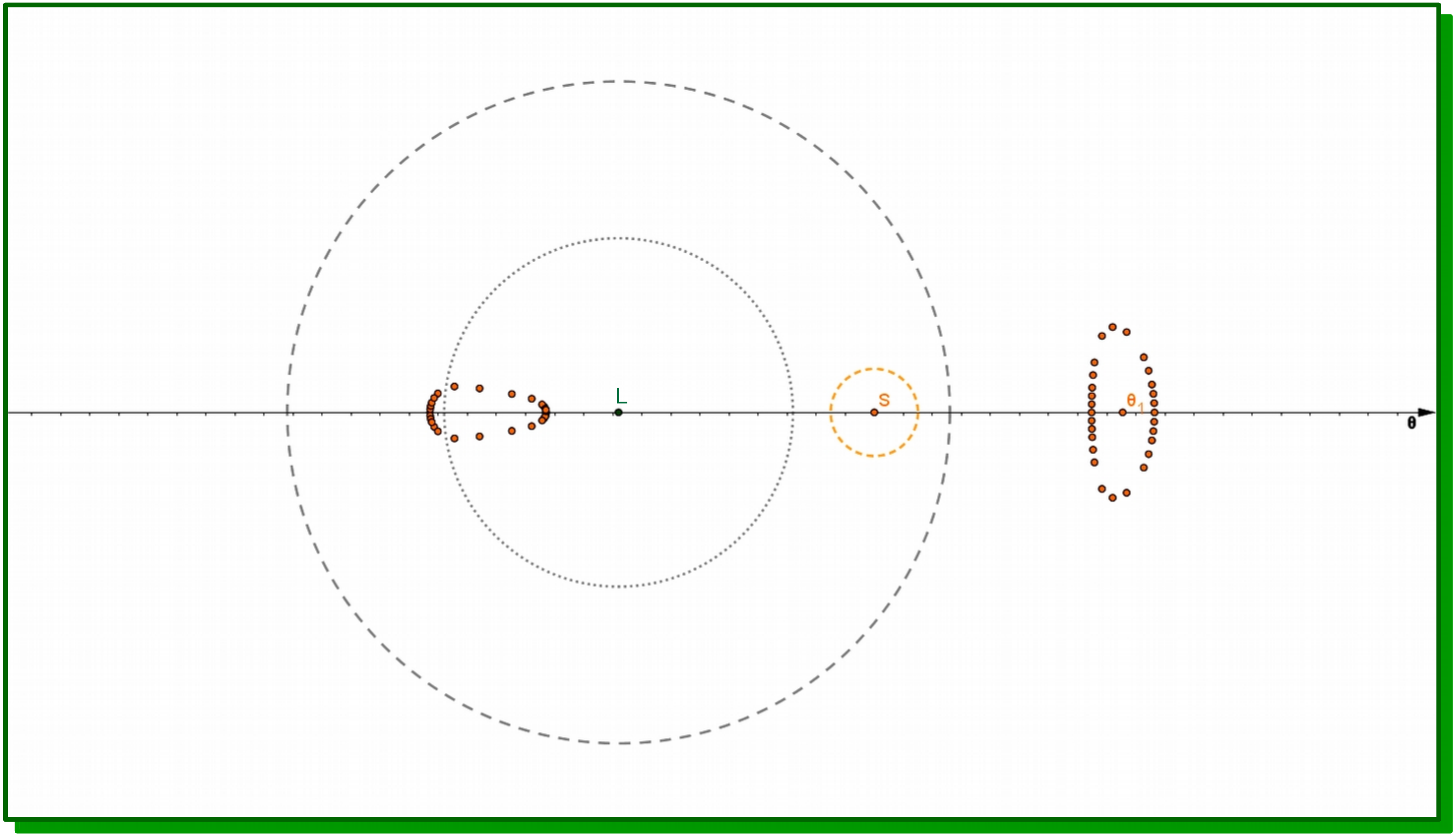

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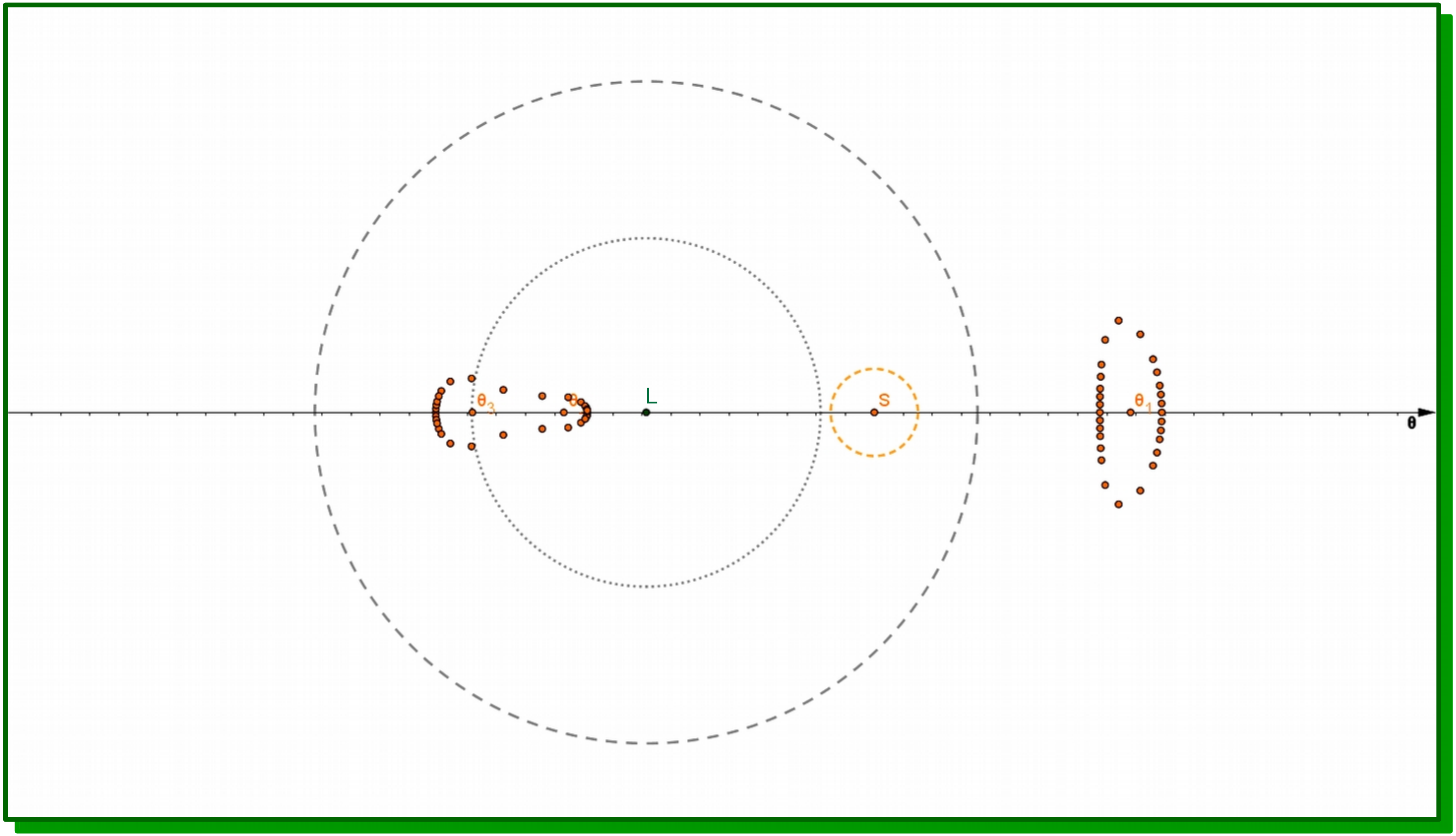

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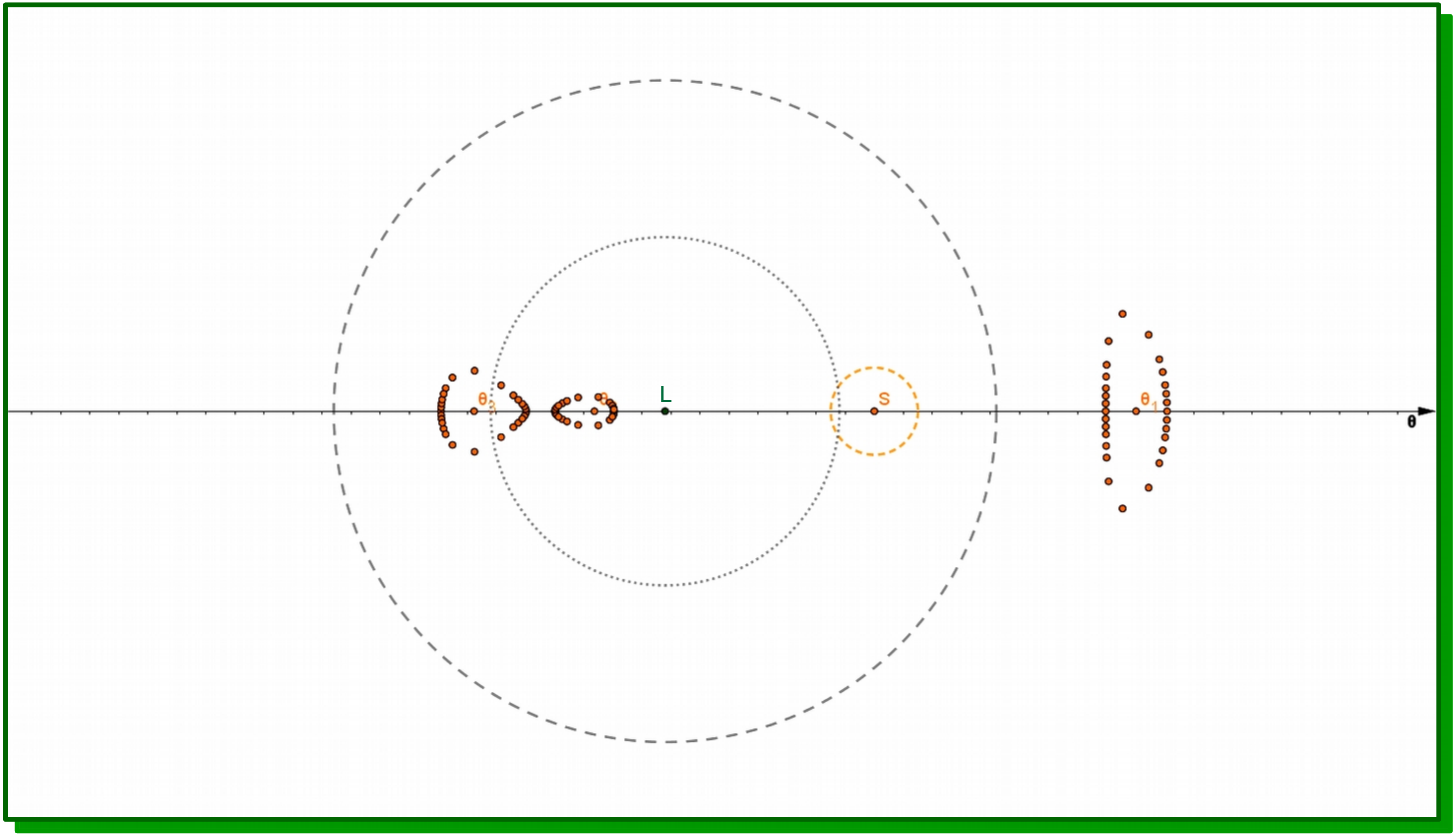

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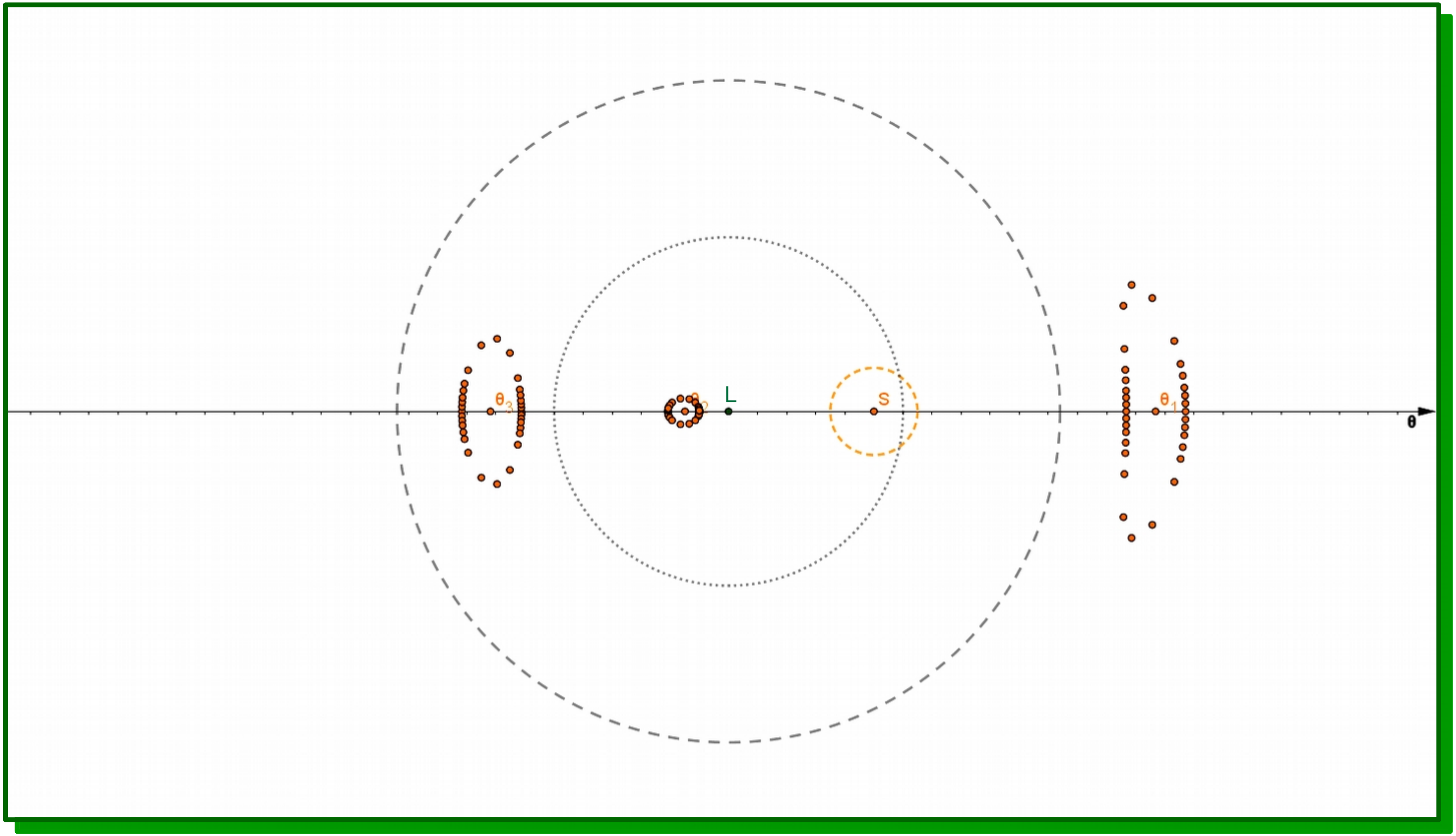

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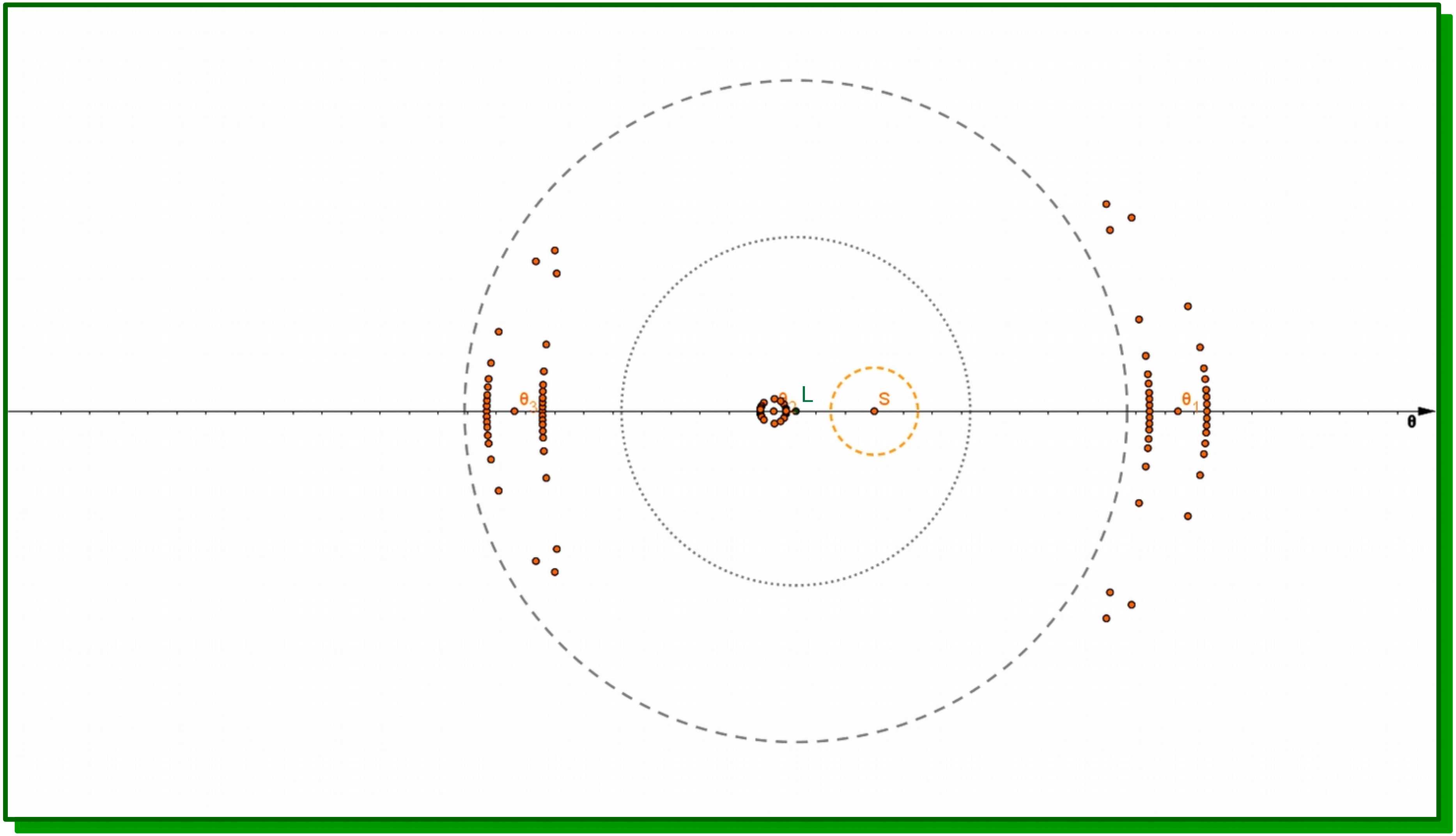

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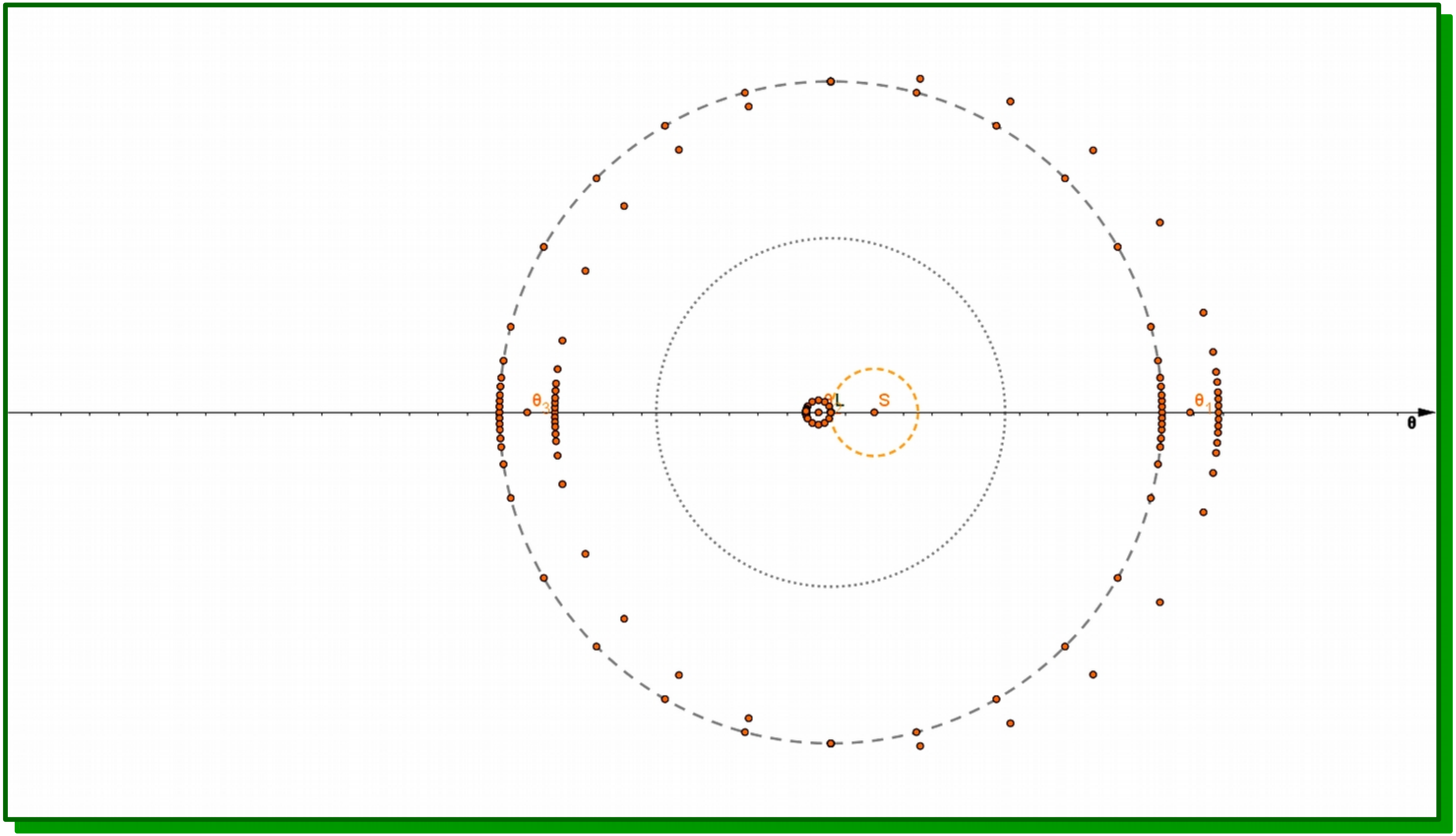

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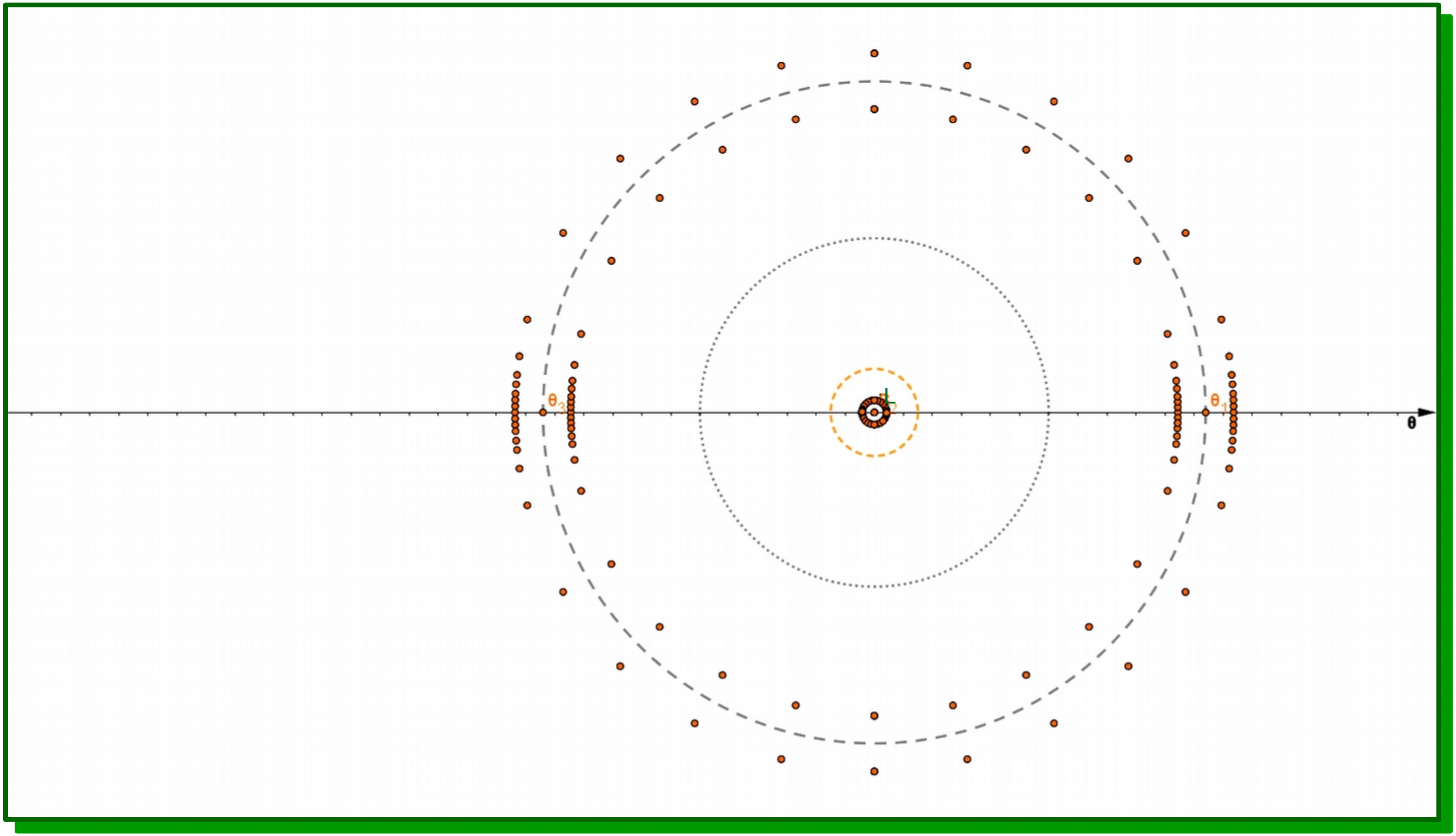

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Other Lenses

- Analyzed and produced:
  - **Point-mass Lens**
  - **Plummer Sphere Lens**
  - **Uniform Disk**
  - **Singular Isothermal Sphere (SIS)**
  - **Kuzmin Disk**

- Only Analyzed:
  - **SIS with a core**
  - **Spiral Galaxy**
  - **Navarro-Frenk-White (NFW)**

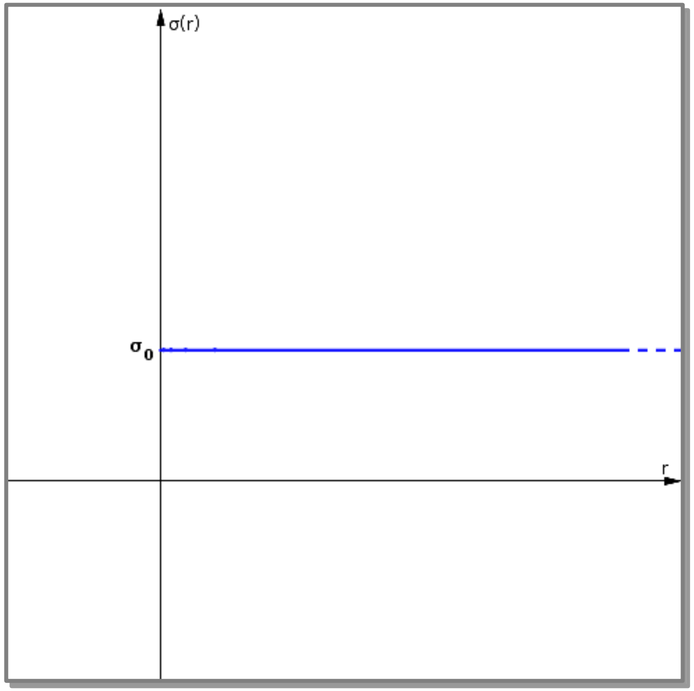
Uniform Disk Lens

- Surface Mass Density

$$\boxed{\sigma(\rho)=\sigma_0=constant}$$

- Projected Mass:

$$\boxed{M(\rho)=\pi\sigma_0\rho^2}$$



$$\theta-\theta_S=\frac{4G}{c^2}\frac{M(\theta)}{\theta}\frac{D_{LS}}{D_L D_S}=\frac{4G}{c^2}\pi\cdot\sigma_0\cdot\theta\frac{D_L D_{LS}}{D_S}$$

$$\theta_S=0 \longrightarrow \frac{4G}{c^2}\pi\cdot\sigma_0\frac{D_L D_{LS}}{D_S}=\frac{\sigma_0}{\Sigma_C}=k=1$$

$$k=\frac{\sigma_0}{\Sigma_C} \rightarrow \begin{array}{lcl} k>1 & & \sigma_0>\Sigma_C \\ k=1 & \rightarrow if & \sigma_0=\Sigma_C \\ k<1 & & \sigma_0<\Sigma_C \end{array}$$

with $\Sigma_C=\frac{c^2}{4G\pi}\frac{D_S}{D_L D_{LS}}$

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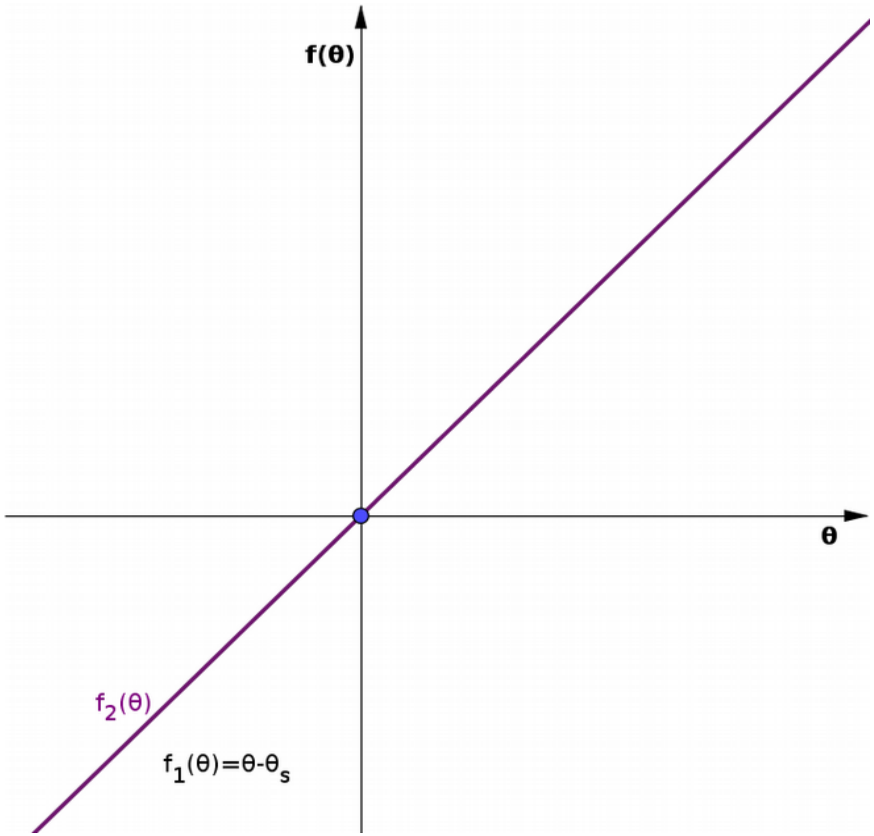
Uniform Disk Lens

$$\boxed{\theta - \theta_s = k \cdot \theta}$$

Lens equation in terms of k

$$\theta_s = 0$$

$$k = 1$$



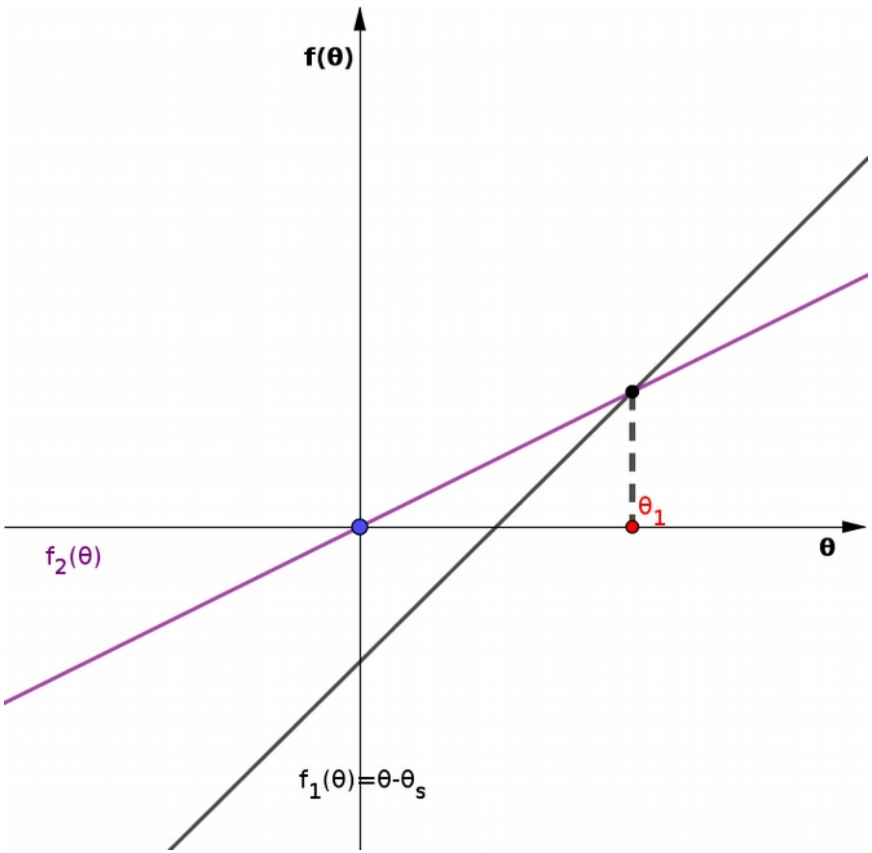
Uniform Disk Lens

$$\boxed{\theta - \theta_s = k \cdot \theta}$$

Lens equation in terms of k

$\theta_s > 0$

$k < 1$



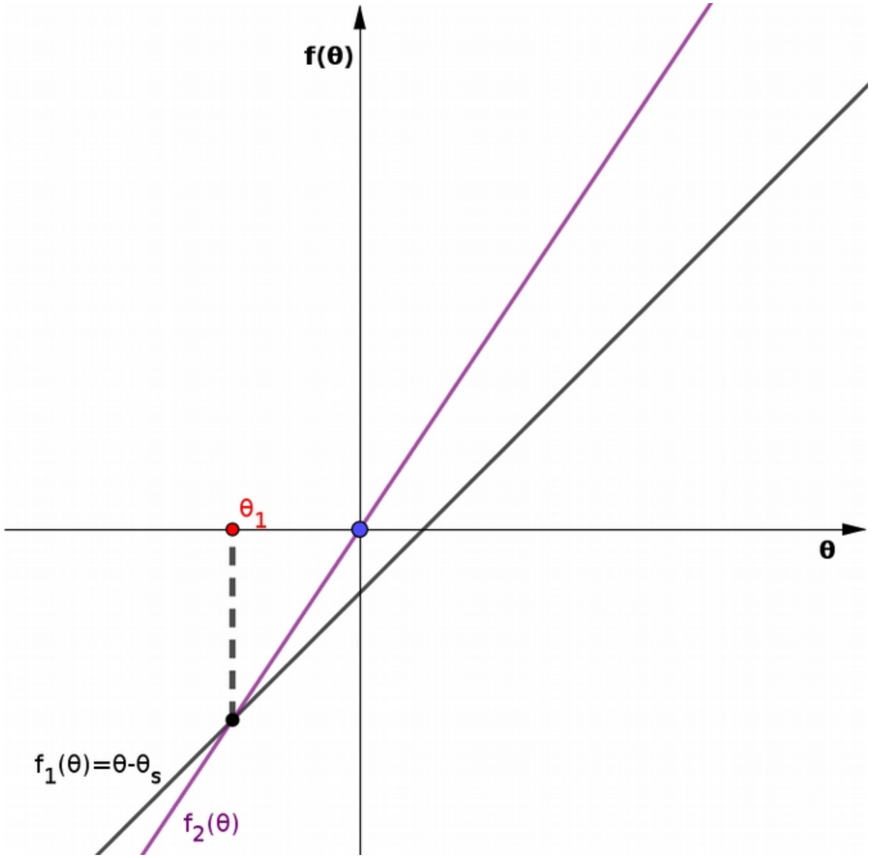
Uniform Disk Lens

$$\boxed{\theta - \theta_S = k \cdot \theta}$$

Lens equation in terms of k

$\theta_S > 0$

$k > 1$



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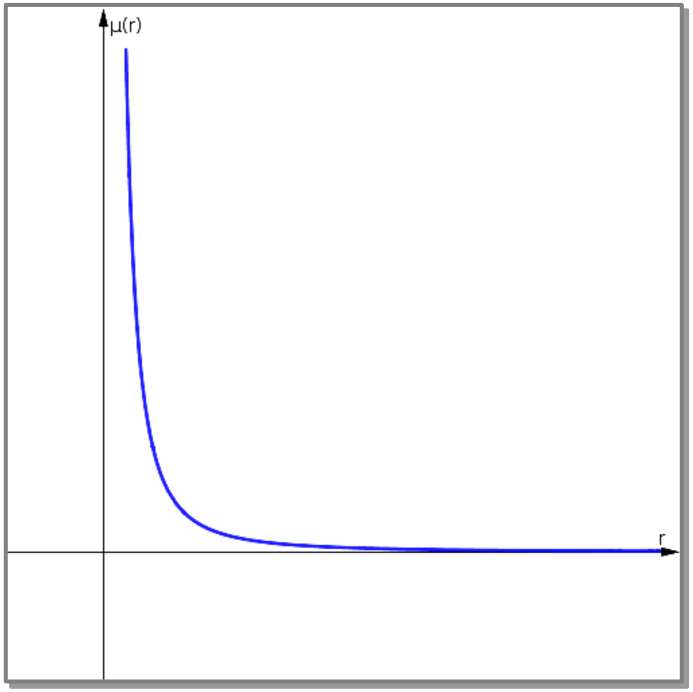
SIS Lens

- Mass Density

$$\boxed{\mu(r)=\frac{\sigma_v^2}{2\pi G r^2}}$$



- Mass:

$$M(r)=\frac{2\sigma_v^2}{G}r \longrightarrow M(\rho)=\frac{\pi\sigma_v^2}{G}\rho$$

Projected Mass

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SIS Lens

$$\theta - \theta_S = \frac{4G}{c^2}\frac{M(\theta)}{\theta}\frac{D_{LS}}{D_L D_S} = \frac{4}{c^2}\pi\sigma_v^2\frac{D_{LS}}{D_S}$$

$$\theta_E = \frac{4}{c^2}\pi\sigma_v^2\frac{D_{LS}}{D_S}$$

$$\boxed{\theta - \theta_S = \theta_E}$$

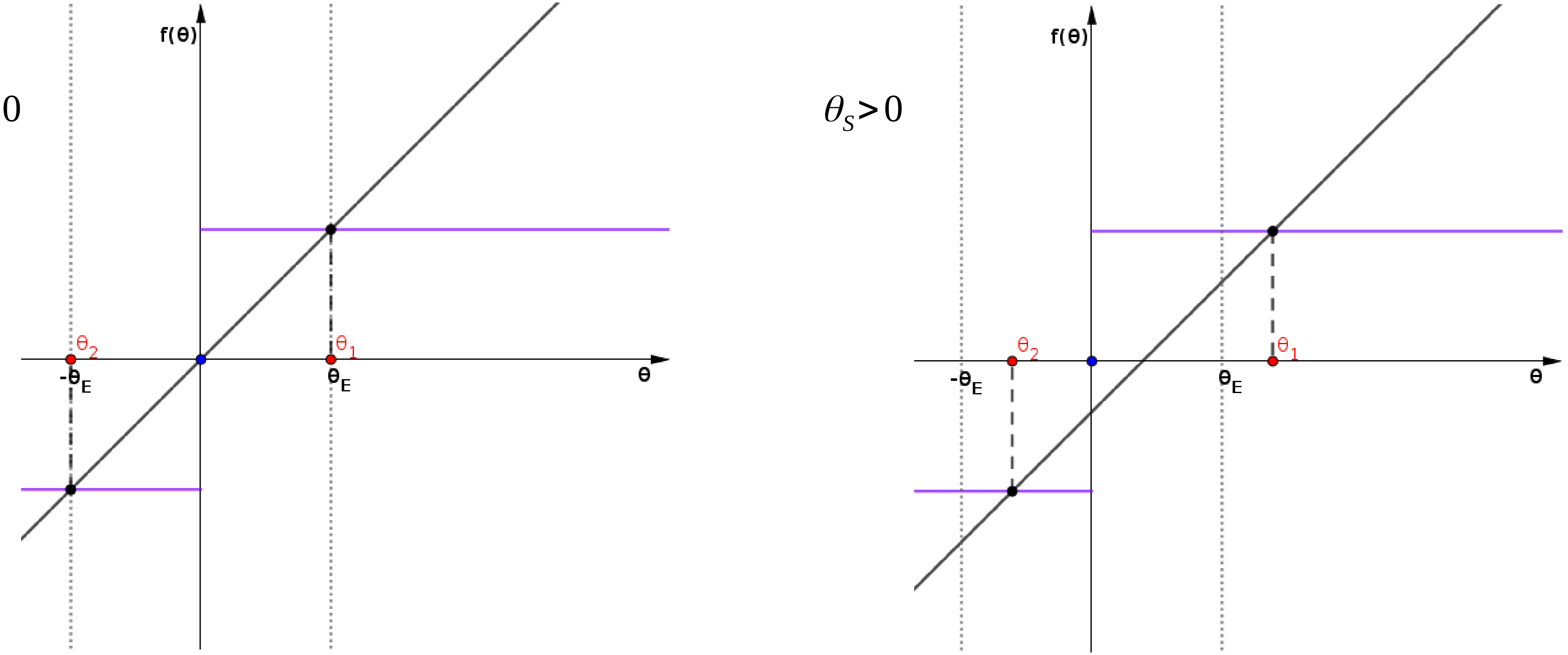
with $sgn(\theta) = sgn(\theta_E)$

Lens equation in terms of Einstein radius

SIS Lens

$$\boxed{\theta - \theta_S = \theta_E}$$

$\theta_S = 0$ $\theta_S > 0$



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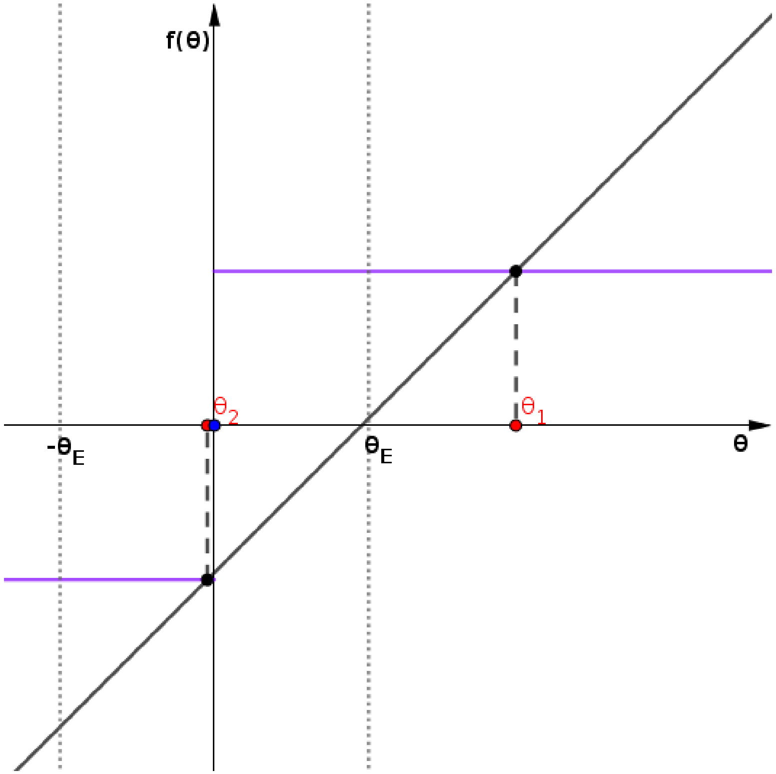
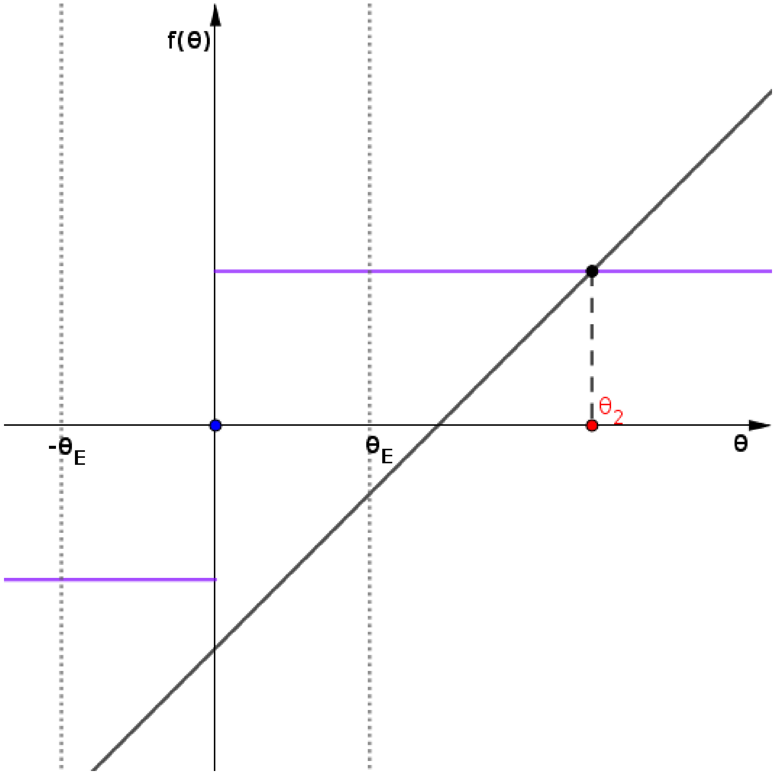
SIS Lens

$$\boxed{\theta - \theta_S = \theta_E}$$

$\theta_S > 0$



$\theta_S > 0$



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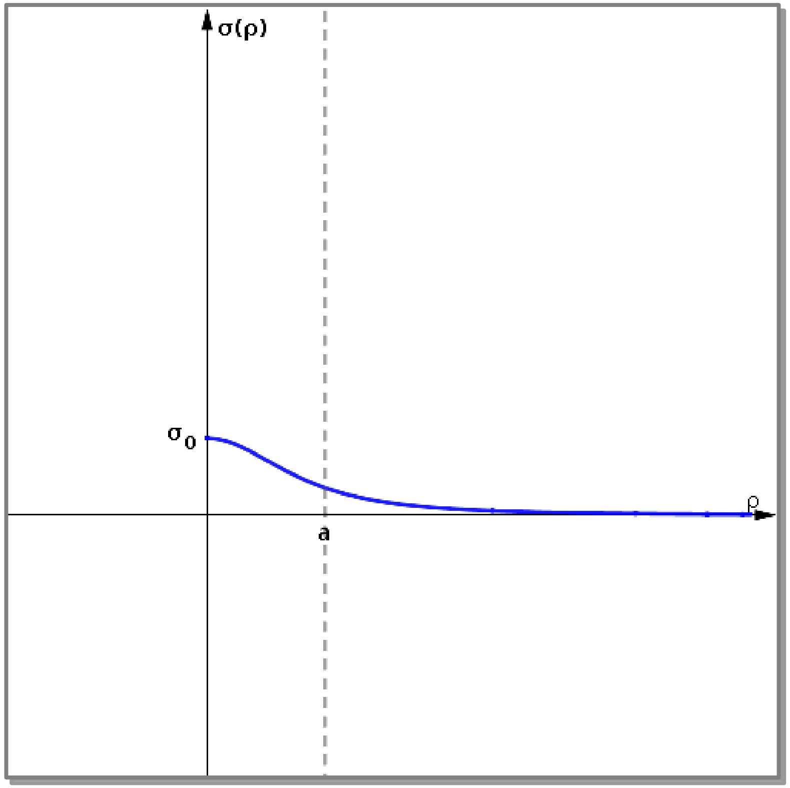
Kuzmin Disk Lens

- Surface Mass Density

$$\boxed{\sigma(\rho)=\frac{a\cdot M}{2\pi}\frac{1}{(\rho^2+a^2)^{3/2}}}$$

- Projected Mass:

$$\boxed{M(\rho)=M\frac{(\rho^2+a^2)^{1/2}-a}{(\rho^2+a^2)^{1/2}}}$$



Kuzmin Disk Lens

$$\theta-\theta_S=\frac{4G}{c^2}\frac{M(\theta)}{\theta}\frac{D_{LS}}{D_L D_S}=\frac{4G}{c^2}\frac{M}{\theta}\frac{(\theta^2+\theta_a{}^2)^{1/2}-\theta_a}{(\theta^2+\theta_a{}^2)^{1/2}}\frac{D_{LS}}{D_L D_S}$$

$$\theta_E=\frac{4G}{c^2}\frac{M(\theta_E)}{\theta_E}\frac{D_{LS}}{D_L D_S}=\frac{4G}{c^2}\frac{M}{\theta_E}\frac{(\theta_E{}^2+\theta_a{}^2)^{1/2}-\theta_a}{(\theta_E{}^2+\theta_a{}^2)^{1/2}}\frac{D_{LS}}{D_L D_S}$$

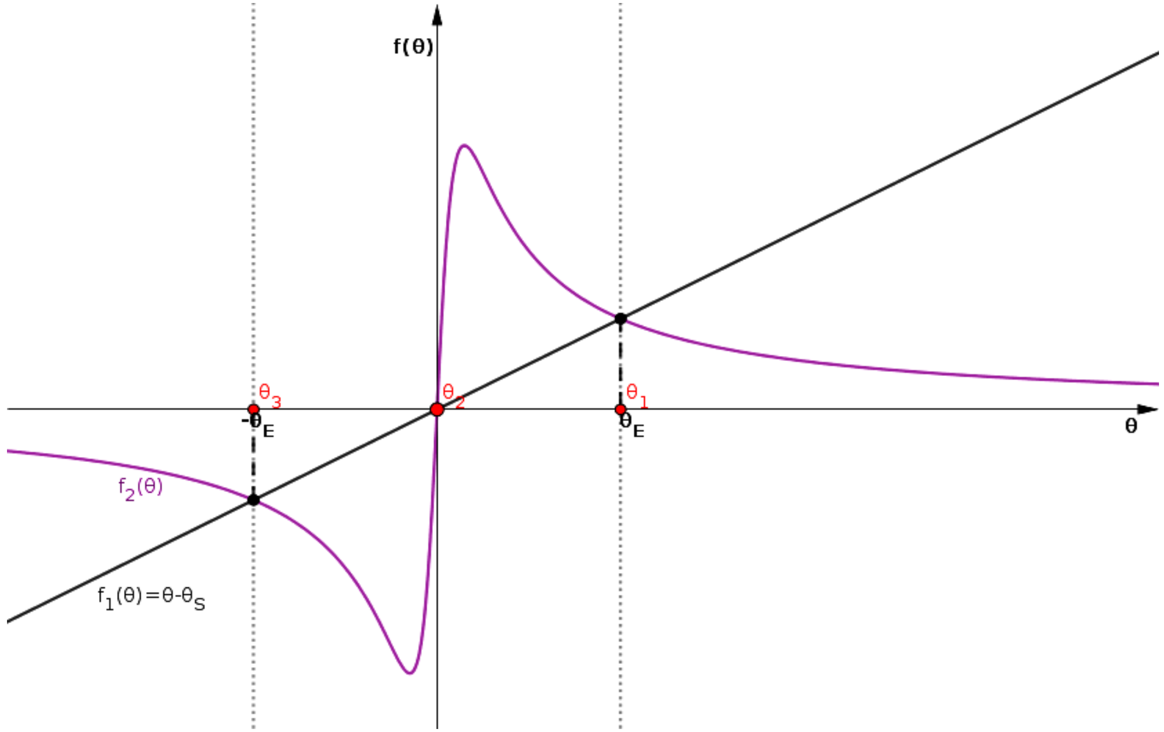
$$\boxed{\theta-\theta_S=\frac{\theta_E{}^2}{\theta}\frac{(\theta_E{}^2+\theta_a{}^2)^{1/2}}{(\theta^2+\theta_a{}^2)^{1/2}}\frac{(\theta^2+\theta_a{}^2)^{1/2}-\theta_a}{(\theta_E{}^2+\theta_a{}^2)^{1/2}-\theta_a}}$$

Lens equation in terms of Einstein radius

Kuzmin Disk Lens

$$\boxed{\theta - \theta_S = \frac{\theta_E^{\;2}}{\theta} \frac{(\theta_E^{\;2} + \theta_a^{\;2})^{1/2}}{(\theta^2 + \theta_a^{\;2})^{1/2}} \frac{(\theta^2 + \theta_a^{\;2})^{1/2} - \theta_a}{(\theta_E^{\;2} + \theta_a^{\;2})^{1/2} - \theta_a}}$$

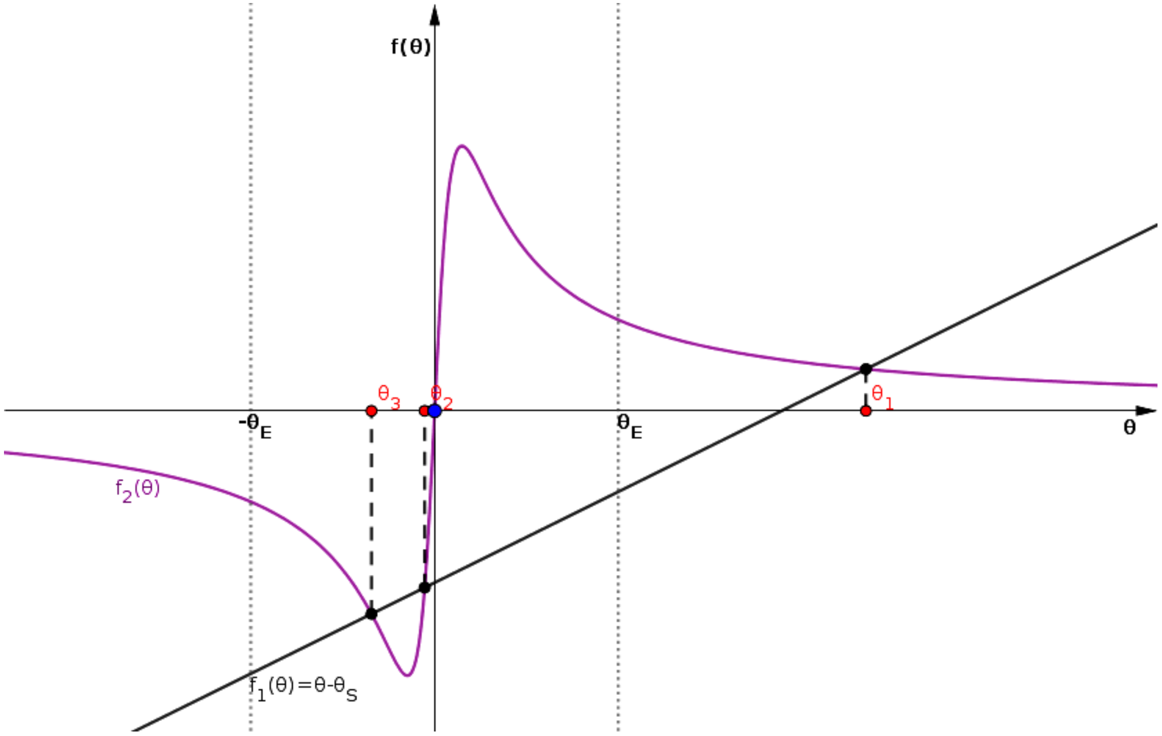
$$\theta_S = 0$$



Kuzmin Disk Lens

$$\theta - \theta_S = \frac{\theta_E^{\ 2}}{\theta} \frac{(\theta_E^{\ 2} + \theta_a^{\ 2})^{1/2}}{(\theta^2 + \theta_a^{\ 2})^{1/2}} \frac{(\theta^2 + \theta_a^{\ 2})^{1/2} - \theta_a}{(\theta_E^{\ 2} + \theta_a^{\ 2})^{1/2} - \theta_a}$$

$\theta_S > 0$



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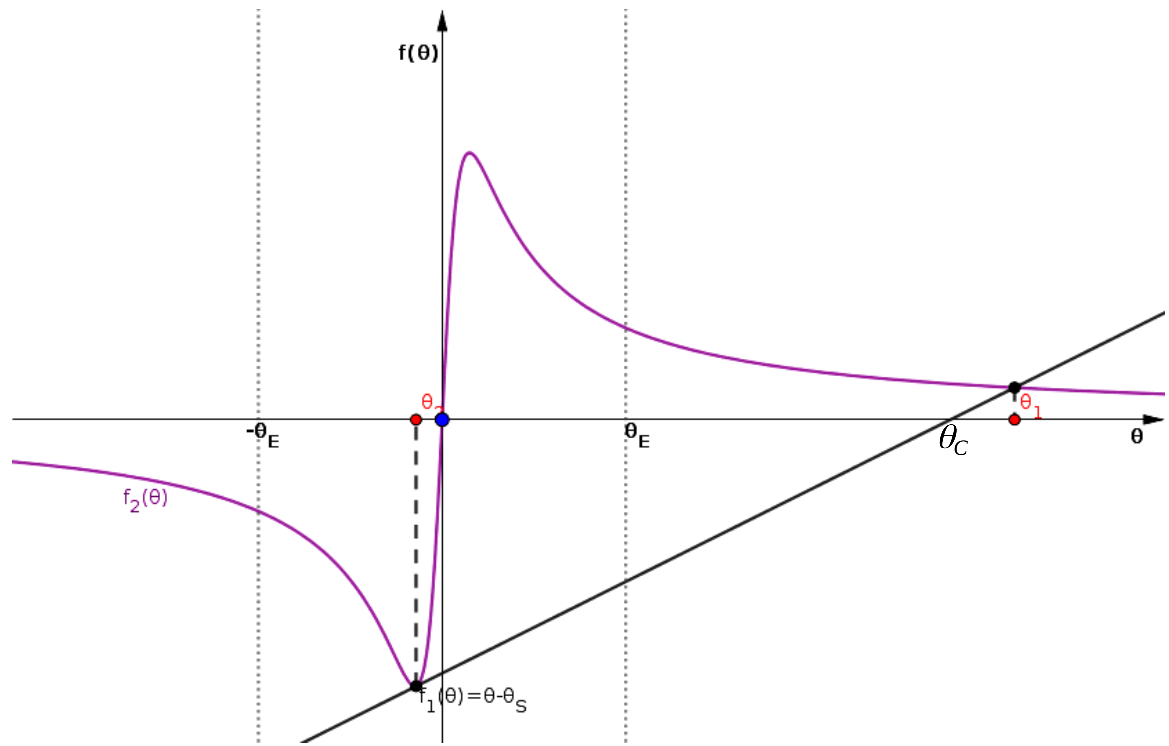
Kuzmin Disk Lens

$$\boxed{\theta-\theta_S=\frac{\theta_E^{\,2}}{\theta}\,\frac{(\theta_E^{\,2}+\theta_a^{\,2})^{1/2}}{(\theta^2+\theta_a^{\,2})^{1/2}}\,\frac{(\theta^2+\theta_a^{\,2})^{1/2}-\theta_a}{(\theta_E^{\,2}+\theta_a^{\,2})^{1/2}-\theta_a}}$$

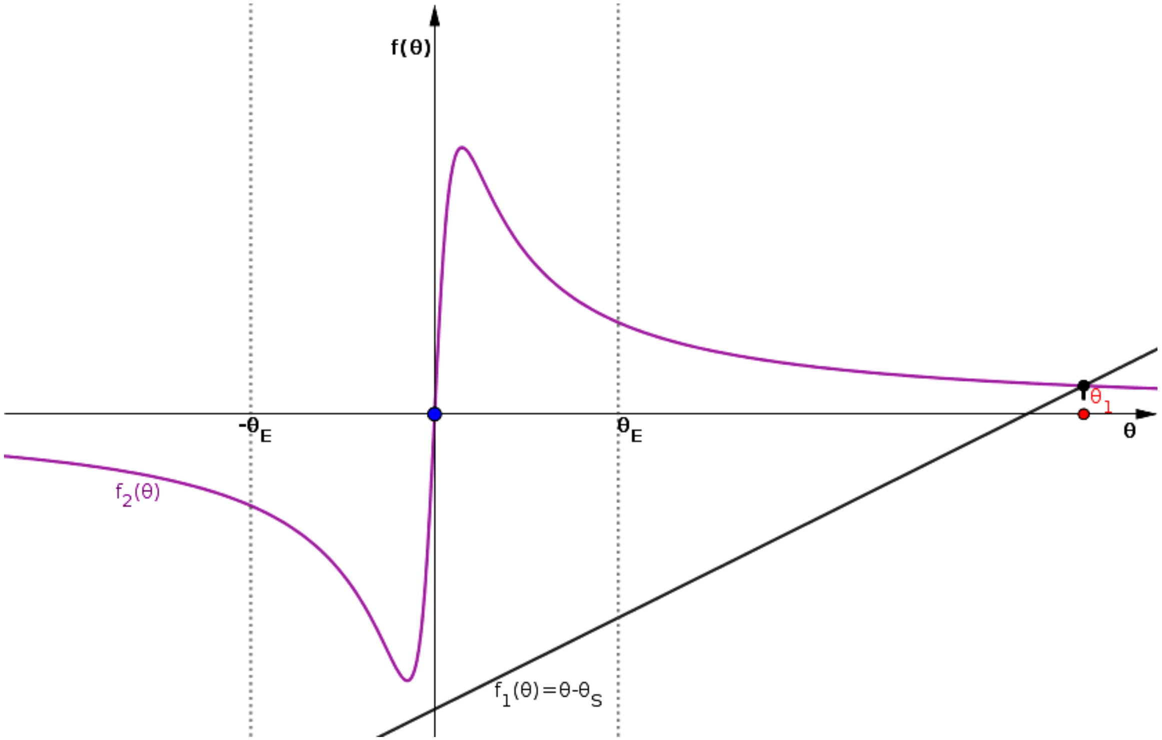
$$\theta_S>0$$
$$\theta_S=\theta_C$$



Kuzmin Disk Lens

$$\boxed{\theta - \theta_S = \frac{\theta_E^{\,2}}{\theta} \frac{\left(\theta_E^{\,2} + \theta_a^{\,2}\right)^{1/2}}{\left(\theta^2 + \theta_a^{\,2}\right)^{1/2}} \frac{\left(\theta^2 + \theta_a^{\,2}\right)^{1/2} - \theta_a}{\left(\theta_E^{\,2} + \theta_a^{\,2}\right)^{1/2} - \theta_a}}$$

$\theta_S > 0$

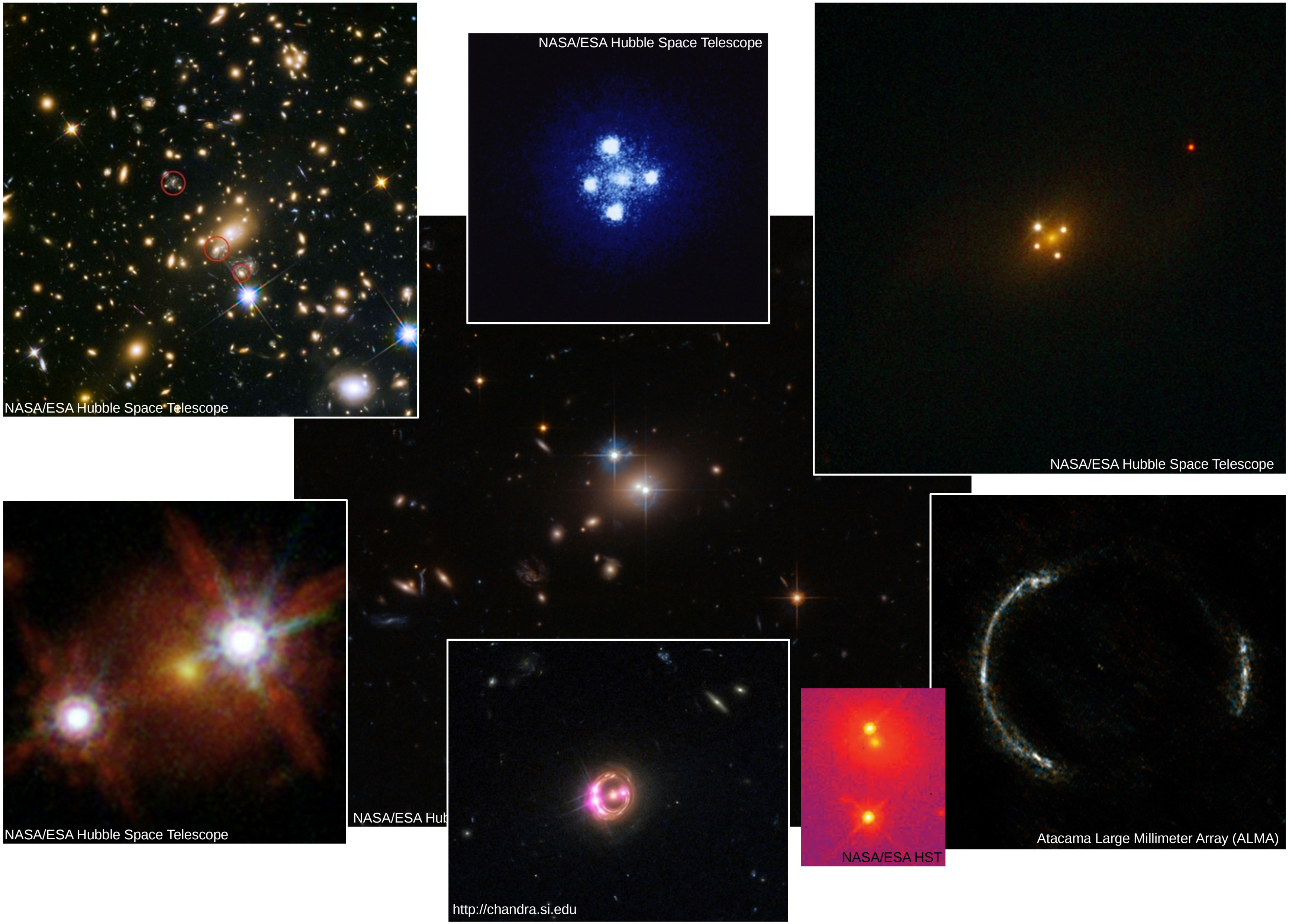


Prof. Dr. Sjur Refsdal

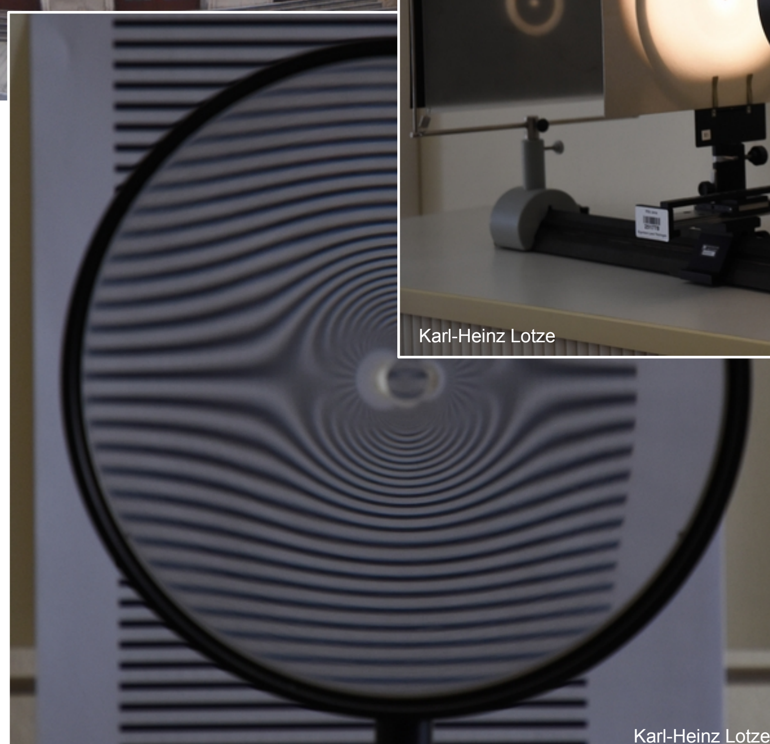
How is it possible to visualize the gravitational lensing effect?

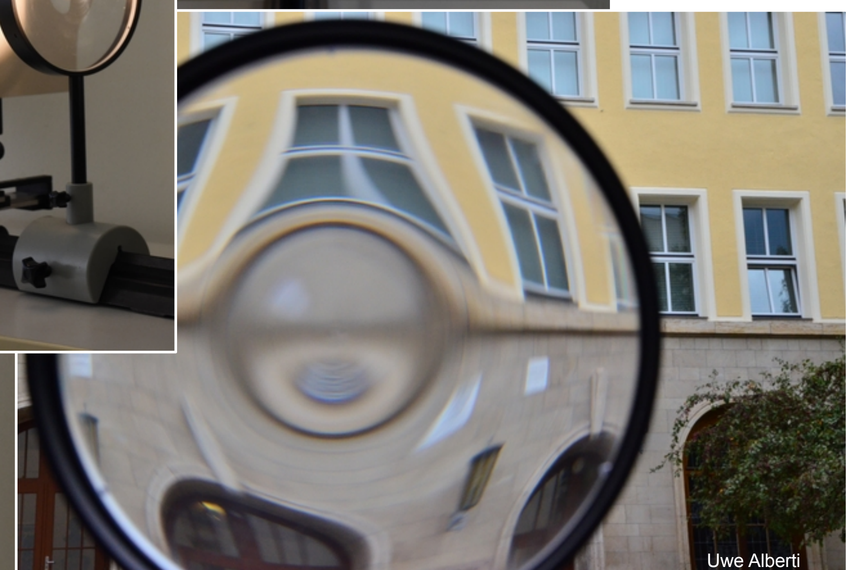
Refsdal, S. and Surdej, J., 1994. Gravitational lenses. Reports on Progress in Physics, 57(2), p.117.

Nowadays this is one possibility



A “gravitational” lens made of glass is another possibility





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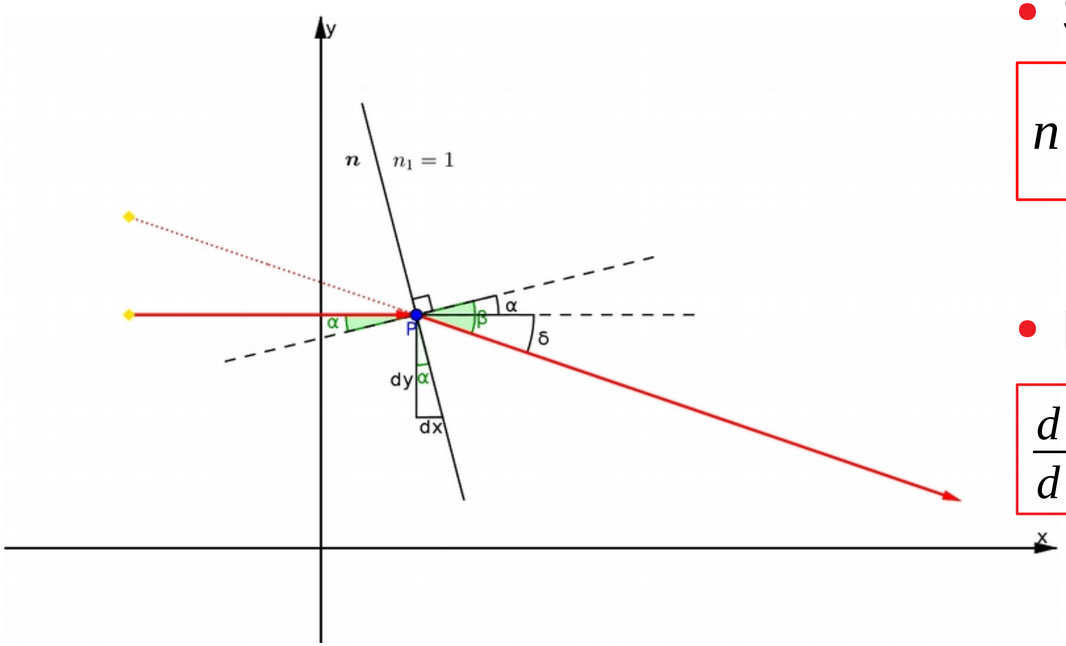
Shaping the Glass Lens



- Snell's Law

$$n = \frac{\sin(\alpha+\delta)}{\sin\alpha} = \cos\delta + \cot\alpha \sin\delta$$

- From figure and Snell's Law

$$\frac{dy}{dx} = -\cot\alpha = -\frac{n-\cos\delta}{\sin\delta} \approx -\frac{n-1}{\delta} = -\frac{(n-1)c^2}{4GM(y)} \cdot y$$

Small-angle Approximation

- Solving this differential equation we obtain the shape of the glass lens

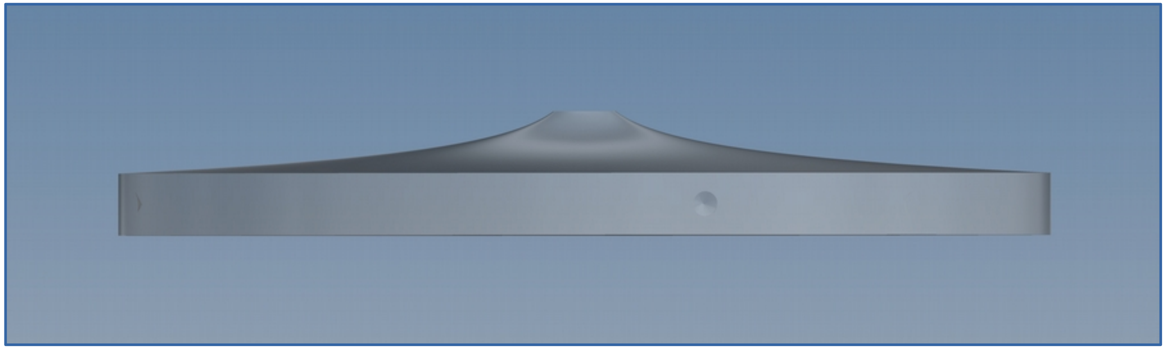
$$\frac{dy}{dx} = -\frac{(n-1)c^2}{4GM(y)} \cdot y$$

Point-Mass Lens

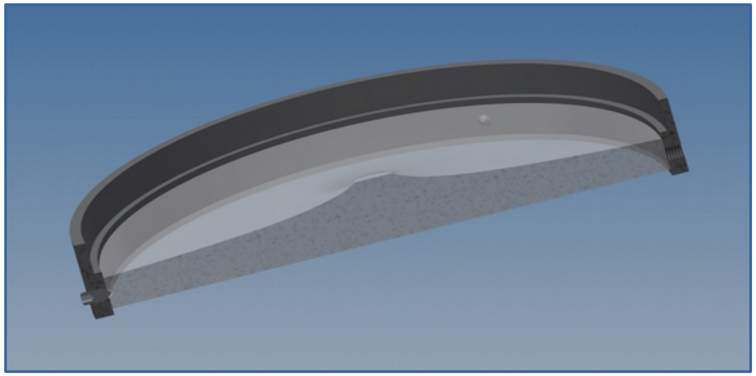
$$\boxed{\frac{dy}{dx} = -\frac{(n-1)c^2}{4GM} \cdot y}$$

- Solution

$$\boxed{y = A\,\mathrm{e}^{-\frac{(n-1)c^2}{4GM} \cdot x}}$$



- Rotational symmetry gives



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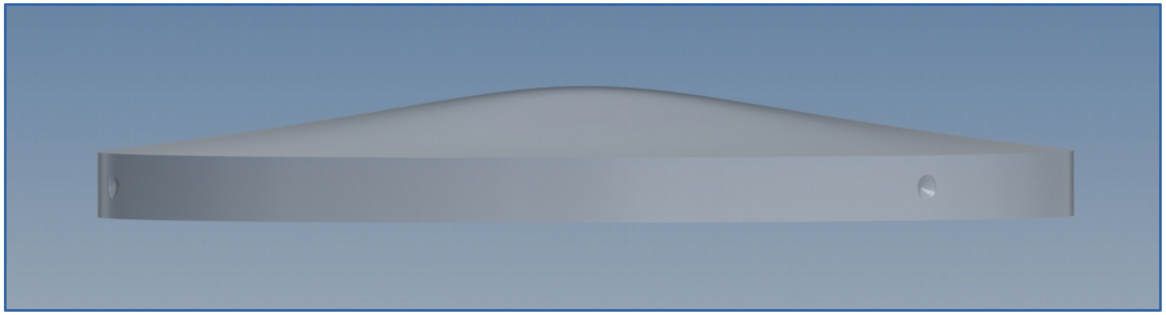
Plummer Sphere Lens

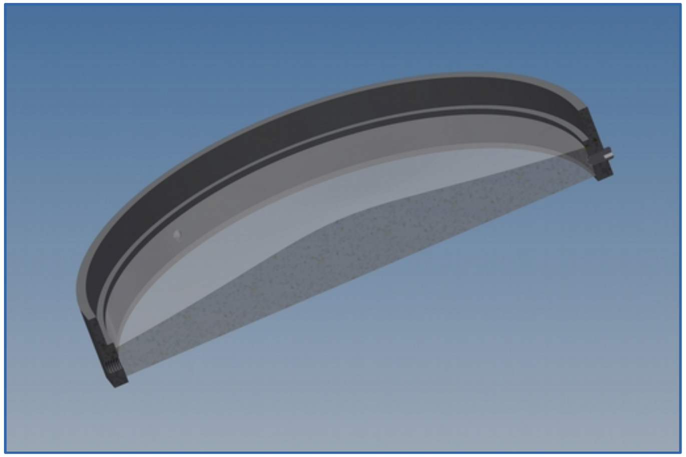
$$\frac{dy}{dx} = -\frac{(n-1)c^2}{4GM}\frac{y^2+a^2}{y}$$

- Solution

$$y = \sqrt{A \cdot \mathrm{e}^{-\frac{(n-1)c^2}{4GM}\cdot x} - a^2}$$



- Rotational symmetry gives

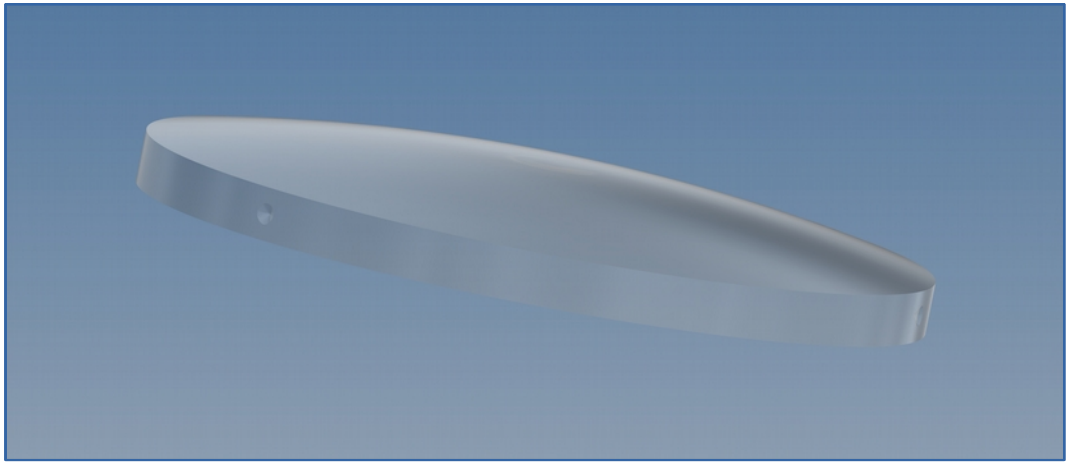


Uniform Disk Lens

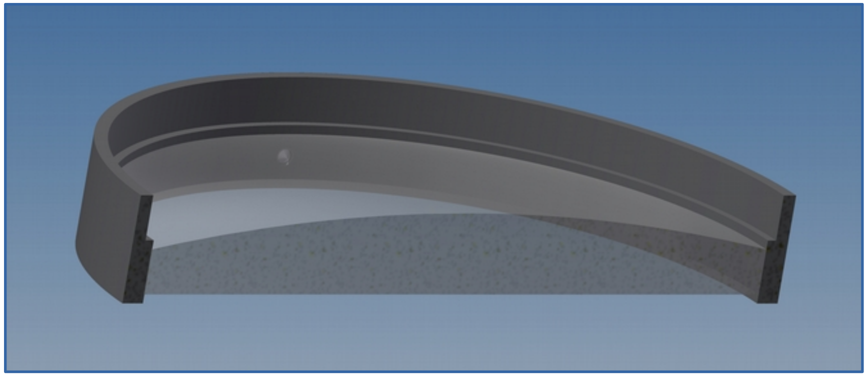
$$\boxed{\frac{dy}{dx} = -\frac{(n-1)c^2}{4G}\frac{1}{\pi\sigma_0 \cdot y}}$$

- Solution

$$\boxed{y = \sqrt{A - 2\frac{(n-1)c^2}{4G\pi\sigma_0} \cdot x}}$$



- Rotational symmetry gives



SIS Lens

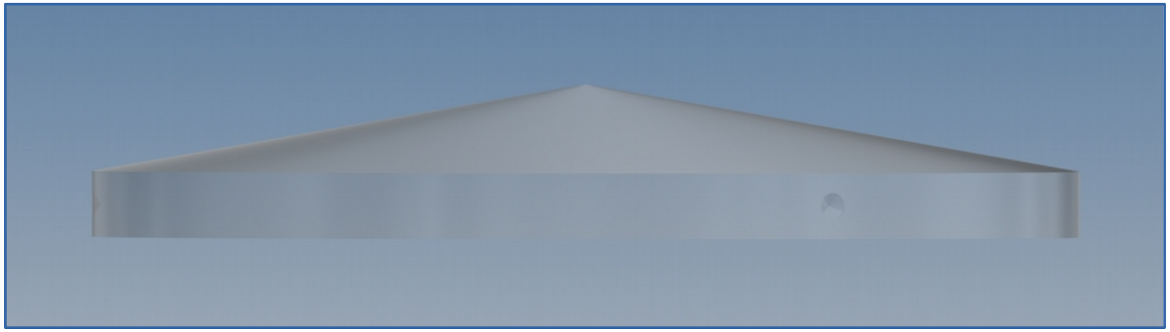
Called also Axicon Lens

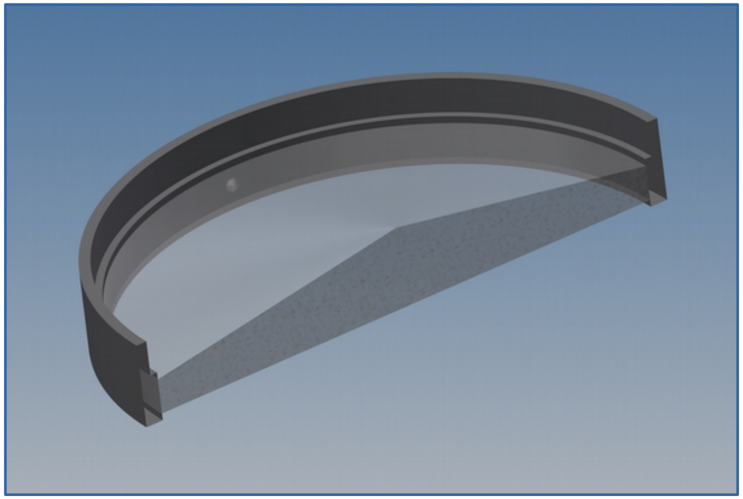
$$\boxed{\frac{dy}{dx} = -\frac{(n-1)c^2}{4}\frac{1}{\pi\sigma_v}}$$

- Solution

$$\boxed{y = A - \frac{(n-1)c^2}{4\pi\sigma_v}\cdot x}$$

- Rotational symmetry gives





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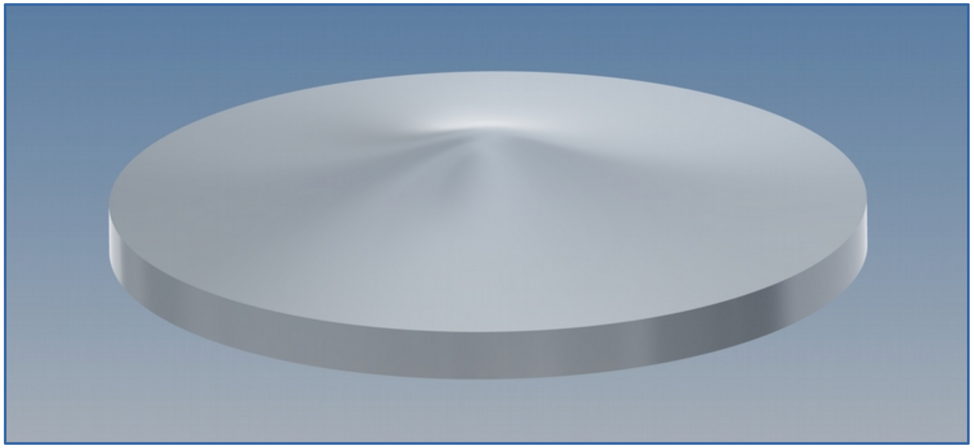
Kuzmin Disk Lens

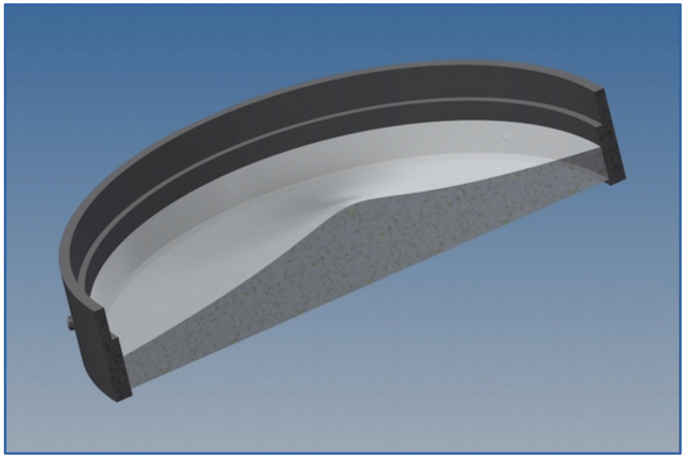
$$\frac{dy}{dx} = -\frac{(n-1)c^2}{4GM}\frac{y(y^2+a^2)^{1/2}}{(y^2+a^2)^{1/2}-a}$$

- Solution

$$y = \sqrt{A \cdot e^{-\frac{(n-1)c^2}{4GM}x}\left(A \cdot e^{-\frac{(n-1)c^2}{4GM}x} - 2a\right)}$$



- Rotational symmetry gives

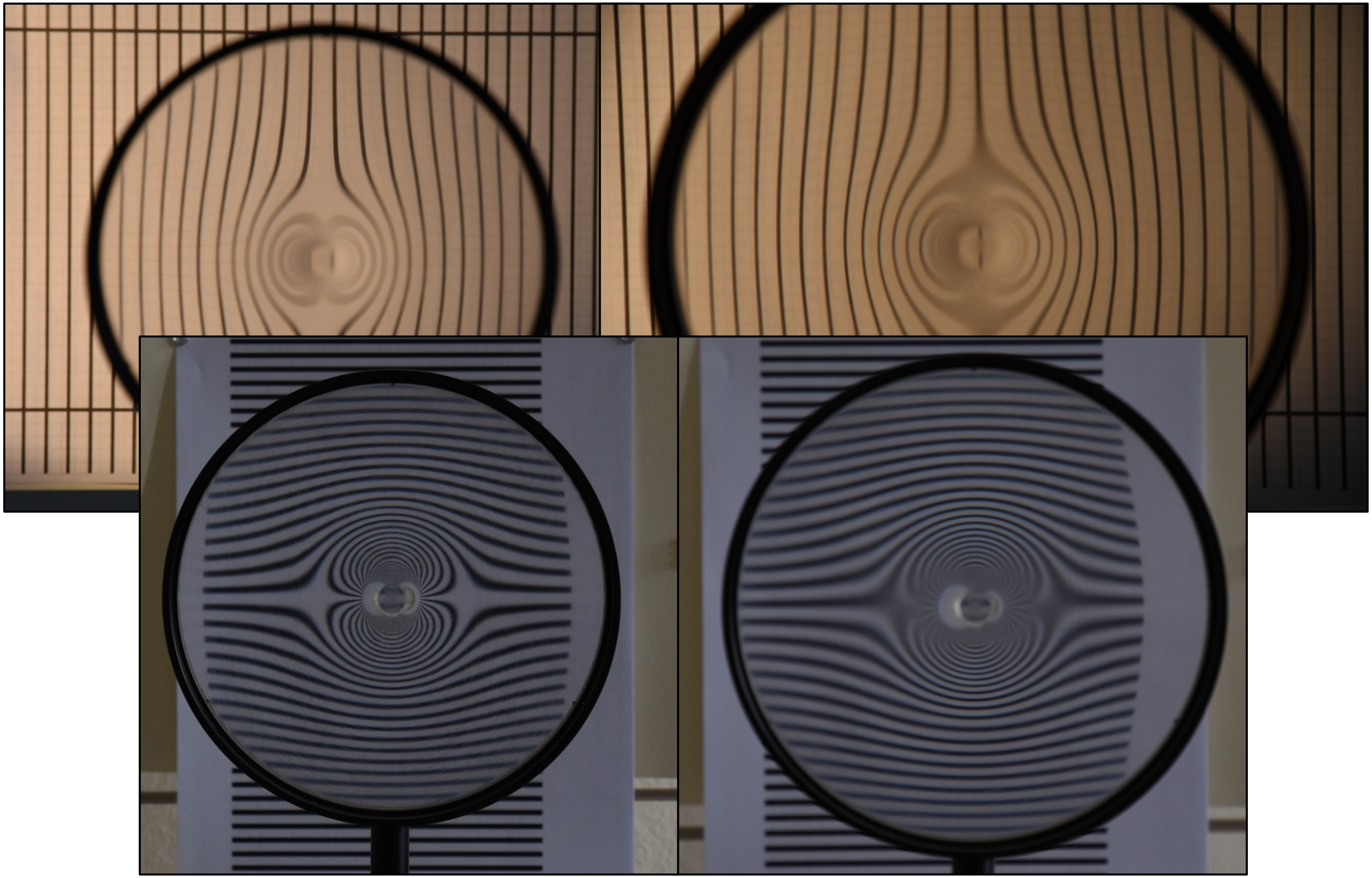


Experimenting

- Of course we already played with the lenses, let's do it together now

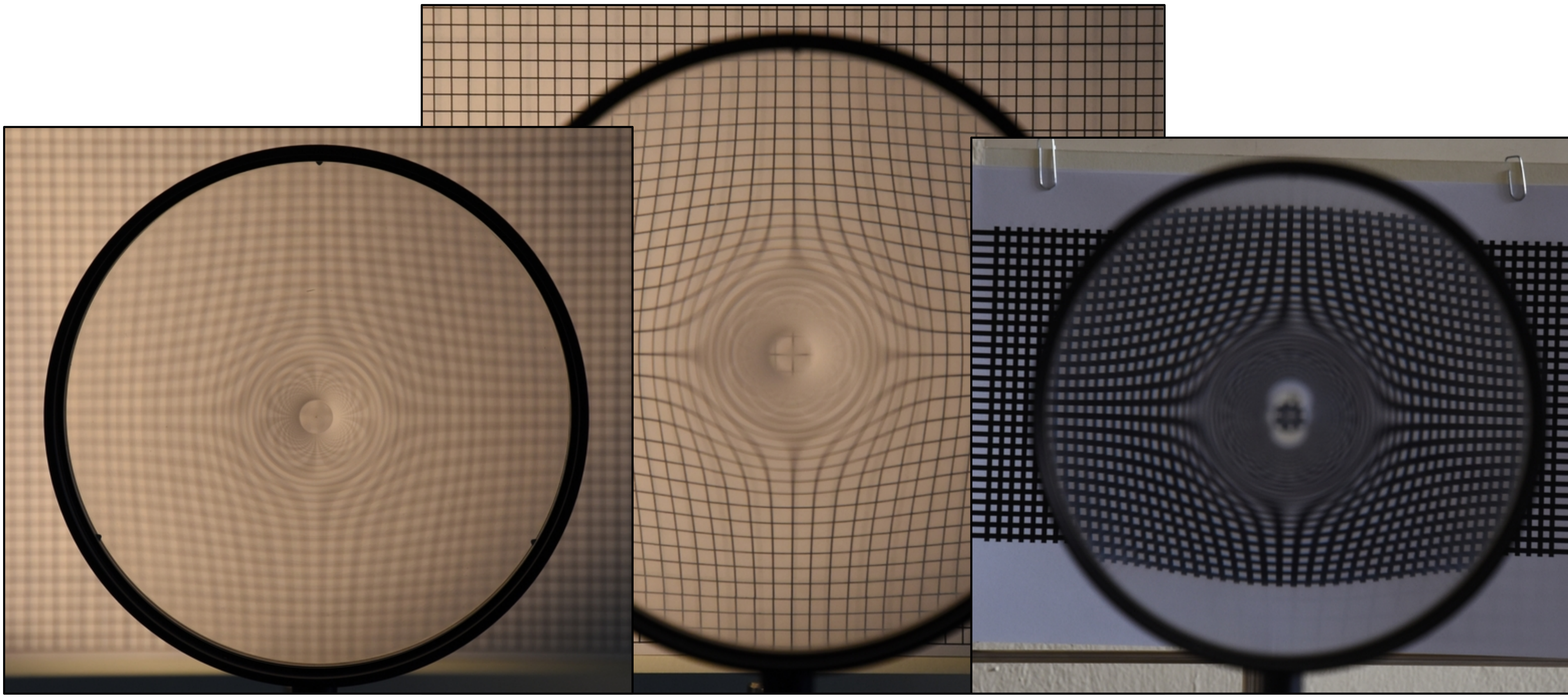
Experimenting

- Lines with the point-mass lens



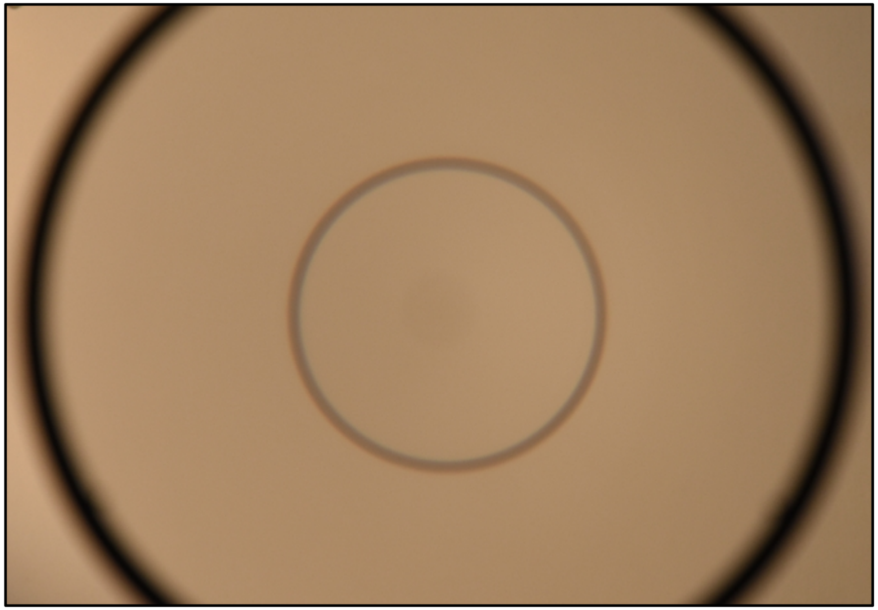
Experimenting

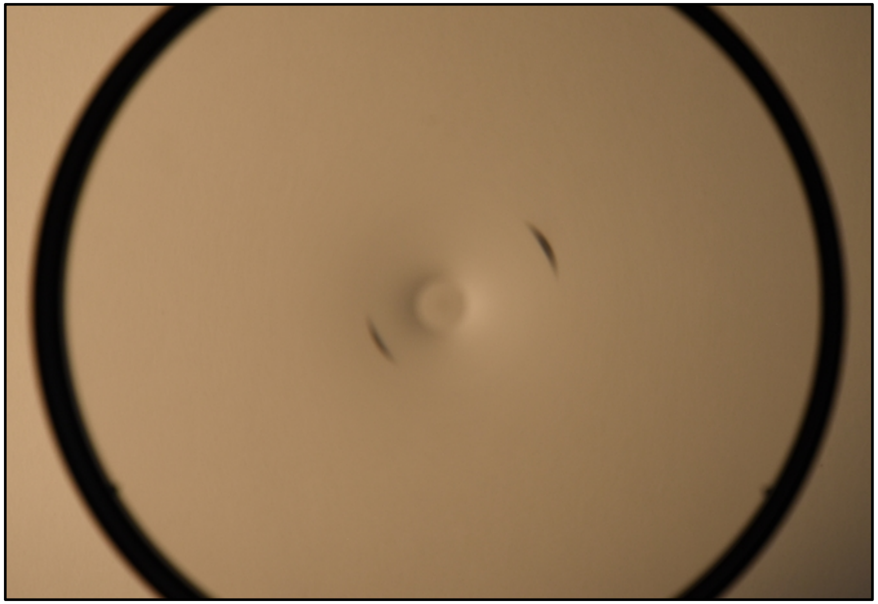
- Grid with the point-mass lens



Experimenting

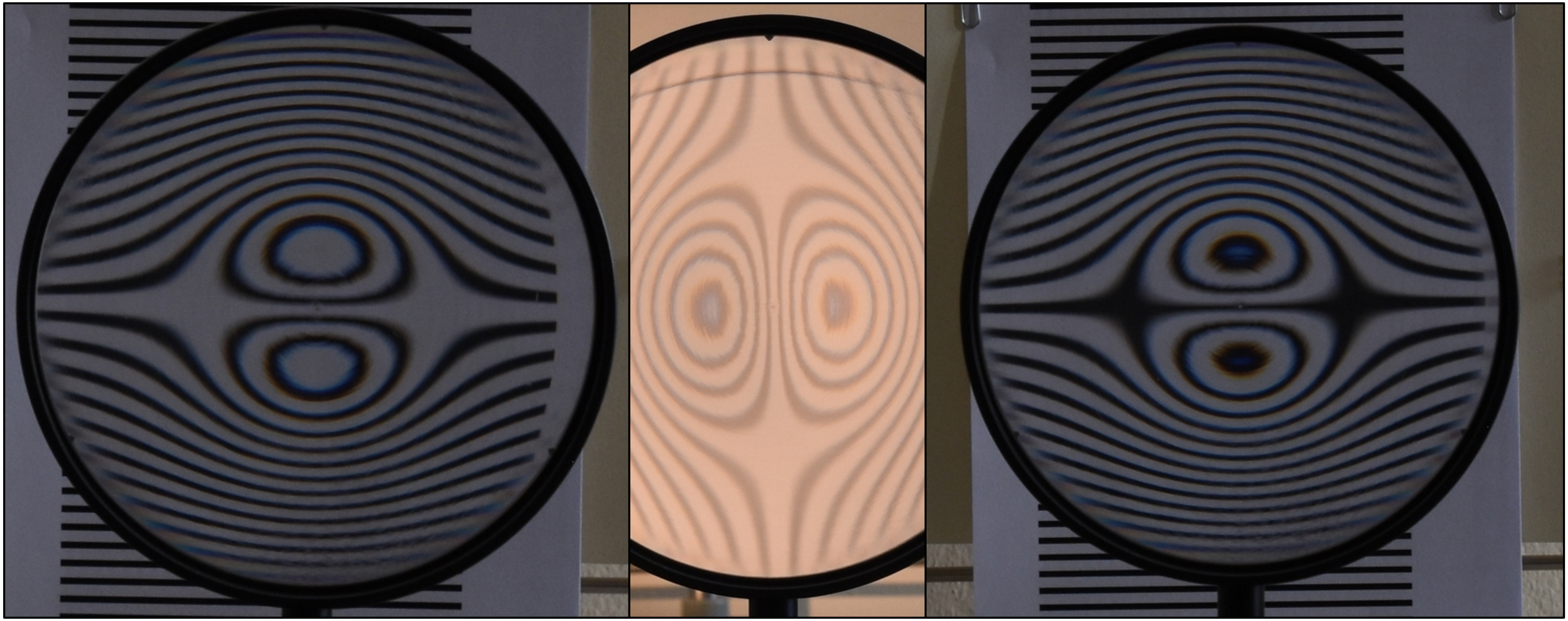
- Extended circular source with the point-mass lens





Experimenting

- Lines with the Plummer lens



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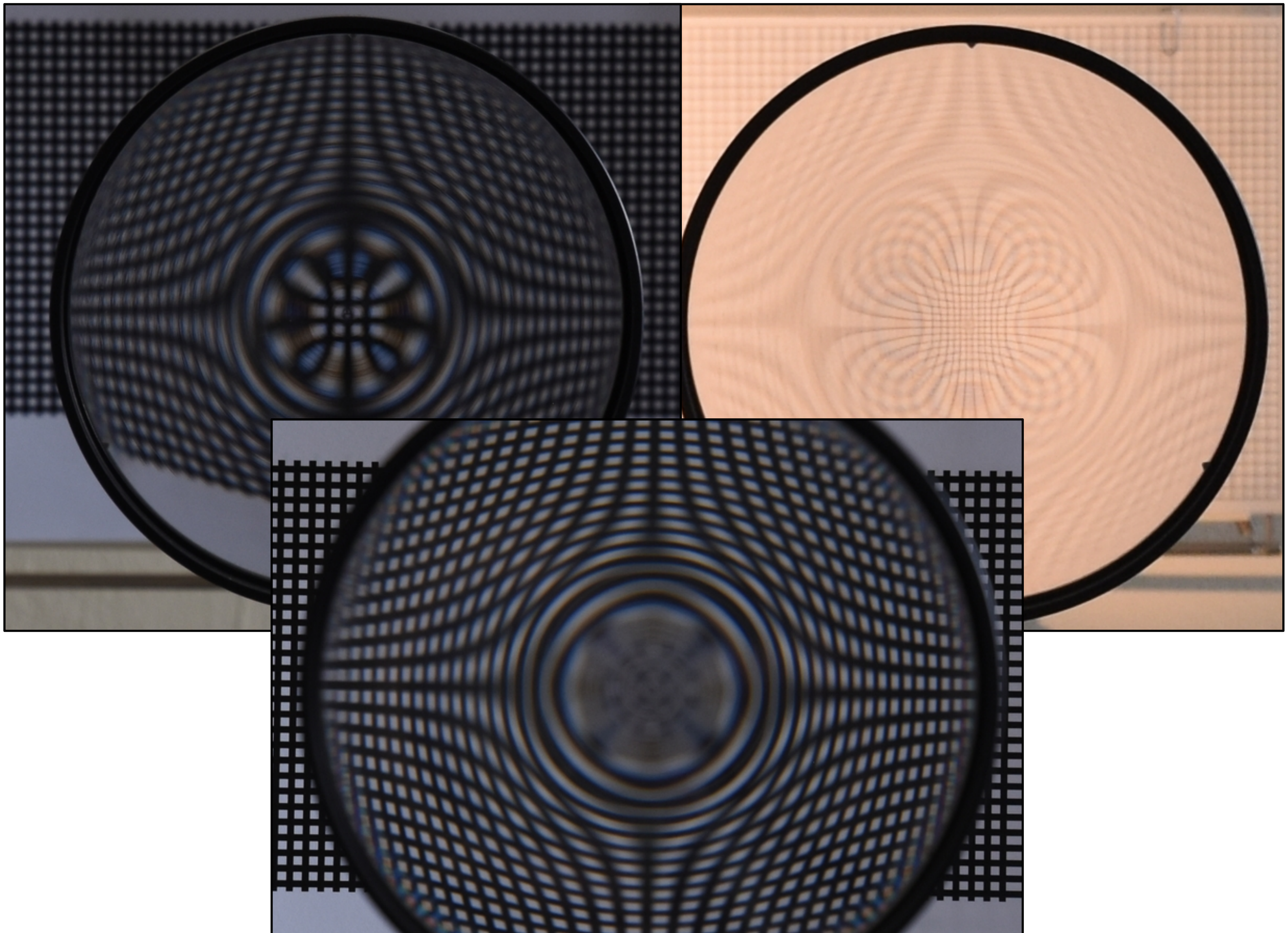
Experimenting

- Grid with the Plummer lens



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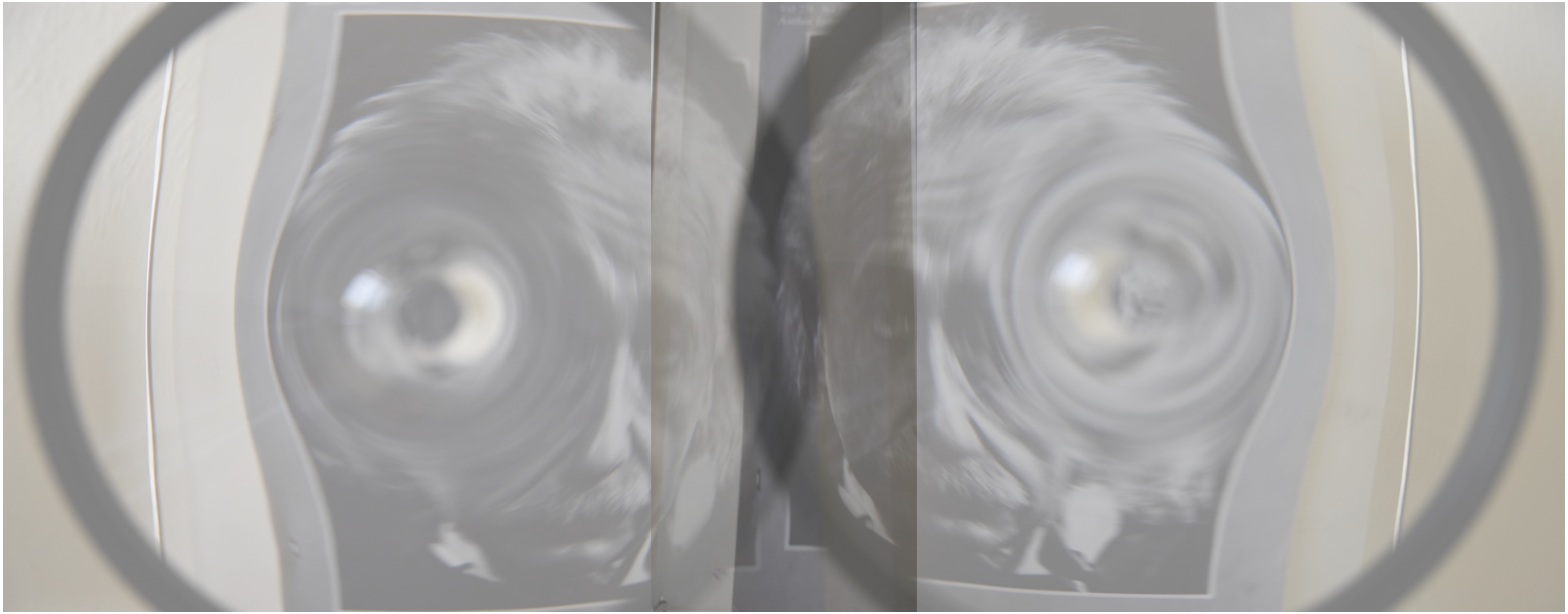
Experimenting

- Extended circular source with the Plummer lens



A big thank to the colleagues who helped and supported us to make what was only on paper in glass:

- Mr Klumbies and his staff from the university lab
- Mr Köhler for the computer design work.

Silvia Simionato